CN35: Orthogonal fncs & "eigenfunctions"

In linear algebra an eigenvalue-eigenvector pair $(\lambda, \underline{x})$ satisfies the equation

$$M\underline{x} = \lambda \underline{x} \quad (\underline{x} \neq \underline{0}),$$

where M is an $n \times n$ matrix.

We can also talk about an eigenvalue-eigenfunction pair (λ, u), which satisfies

$$Lu = \lambda u \quad (u \neq 0),$$

where L is a linear operator such as a linear differential operator. For example for $L = -\frac{d^2}{dx^2}$,

$$-\frac{d^2}{dx^2}u = \lambda u$$

has solutions $u = \sin(\sqrt{\lambda}x)$, $u = \cos(\sqrt{\lambda}x)$, $u = e^{i\sqrt{\lambda}x}$.

Check this!

Symmetric matrices have orthogonal eigenvectors.

Symmetric linear operators have orthogonal eigenfunctions.

The Legendre polynomials can be found by obtaining the eigenfunctions of the differential operator L such that

$$Lu = \frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right].$$ (Solve $Lu = \lambda u$.)

The double Fourier series comes from the DE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda u.$$

The Chebyshev polynomials come from the weighted differential eq. $\frac{d}{dx}\left[w^{-1}\frac{du}{dx}\right] = \lambda w u$,

$$w = \frac{1}{\sqrt{1-x^2}}.$$

Different weights give other sets of orthogonal functions:

$$e^{-x} \rightarrow \text{Laguerre polynomials}$$

$$e^{-x^2/2} \rightarrow \text{Hermite polynomials}$$

Recall from our Fourier-series derivation how we expand a fnc $f(x)$ into a linear combination of orthogonal functions $T_0, T_1, T_2, \ldots$:

We want $f(x) = c_0 T_0 + c_1 T_1 + c_2 T_2 + \ldots$

The problem is to find each coefficient c_k.

The method is to multiply the eq. by $T_k(x)$ & by the weight function $w(x)$.

$$f T_k w = c_0 T_0 T_k w + c_1 T_1 T_k w + c_2 T_2 T_k w + \cdots + c_k T_k T_k w + \cdots$$

Then integrate over the interval on which the functions T_k are orthogonal:

$$\int f T_k w \, dx = c_0 \int T_0 T_k w \, dx + \cdots + c_k \int T_k^2 w \, dx + \cdots.$$

Because of the orthogonality, all the terms on the right are zero except for one (just like for Fourier series).

$$\int f T_k w \, dx = c_k \int T_k^2 w \, dx$$

$$\Rightarrow \boxed{c_k = \frac{\int f T_k w \, dx}{\int T_k^2 w \, dx} \text{ in the weighted orthogonal expansion } f = c_0 T_0 + c_1 T_1 + c_2 T_2 + \cdots}$$