CN 34: W ghted inner products and more orthogonal functions

Let's define an inner product on vector space of functions as

$$\langle f, g \rangle = \int_{a_1}^{a_2} f(x)\, g(x)\, w(x)\, dx,$$ weighted inner product with positive weight w.

where $w(x)$ is a given positive function on $a_1 < x < a_2$.

Demonstrate that this defines an inner product on a function space S over $\mathbb{R}$.

$\langle f, g \rangle$ defines a function that assigns to each ordered pair of elements $f, g \in S$ a scalar $\langle f, g \rangle \in \mathbb{R}$ such that for all $f, g, h \in S$ and all scalars $\alpha \in \mathbb{R}$, four properties hold:

① $\langle f+g, h \rangle \stackrel{?}{=} \langle f, h \rangle + \langle g, h \rangle$.

$$\langle f+g, h \rangle = \int_{a_1}^{a_2} (f+g)(x)\, h(x)\, w(x)\, dx$$

$$= \int_{a_1}^{a_2} \big(f(x)+g(x)\big) h(x) w(x)\, dx$$

$$= \int_{a_1}^{a_2} \big(f(x)h(x)w(x) + g(x)h(x)w(x)\big)\, dx$$

$$= \int_{a_1}^{a_2} f(x)h(x)w(x)\, dx + \int_{a_1}^{a_2} g(x)h(x)w(x)\, dx$$

$$= \langle f, h \rangle + \langle g, h \rangle \checkmark$$

② $\langle \alpha f, g \rangle \stackrel{?}{=} \alpha \langle f, g \rangle$

$$\langle \alpha f, g \rangle = \int_{a_1}^{a_2} (\alpha f)(x)\, g(x)\, w(x)\, dx$$

$$= \int_{a_1}^{a_2} \alpha f(x) g(x) w(x)\, dx$$

$$= \alpha \int_{a_1}^{a_2} f(x) g(x) w(x)\, dx$$

$$= \alpha \langle f, g \rangle \checkmark$$

(3) $\langle g, f \rangle \stackrel{?}{=} \overline{\langle f, g \rangle}$

$$\langle g, f \rangle = \int_{a_1}^{a_2} g(x) f(x) w(x)\, dx$$

$$= \overline{\overline{\int_{a_1}^{a_2} f(x) g(x) w(x)\, dx}}$$

$$= \overline{\int_{a_1}^{a_2} f(x) g(x) w(x)\, dx}$$

because $\int_{a_1}^{a_2} f(x) g(x) w(x)\, dx$ is real

$$= \overline{\langle f, g \rangle} \quad ✓$$

(4) $\langle f, f \rangle > 0$ if $f \neq 0$

$$\langle f, f \rangle = \int_{a_1}^{a_2} [f(x)]^2 w(x)\, dx > 0$$

if f is not the zero function.

$w(x) > 0$, & "area under a curve" that lies above the x-axis is positive.

p278 How can we find other sets of orthog. funcs? More on orthog, 4

(1) We could ~~start~~ with powers of x : $1, x, x^2, \ldots$ and force them to be orthogonal with respect to a given inner product by subtracting from each x^k its projection onto the polynomials of lower degree that have already been "orthogonalized." This is Gram-Schmidt orthogonalization". It produces the Legendre polynomials

$$P_0 = 1, \quad P_1 = x, \quad P_2 = x^2 - \tfrac{1}{3}, \ldots$$

(2) We could invent examples like the Walsh funcs Fig 4.4 of Strang's book.

(3) We could combine sin kx with sin ly into a double Fourier sine series :

$$f(x,y) = b_{11} \sin x \sin y + b_{21} \sin 2x \sin y + b_{12} \sin x \sin 2y + \cdots$$

This applies to a func f that is odd & periodic in both x & y.

Strang, Introduction to [more orthogs]
Applied Mathematics

gonalized) polynomials of lower degree. This is the Gram-Schmidt idea, and it produces the ***Legendre polynomials*** $P_0 = 1$, $P_1 = x$, $P_2 = x^2 - \frac{1}{3}$, ...

(2) We could invent examples like the ***Walsh functions*** in Fig. 4.4:

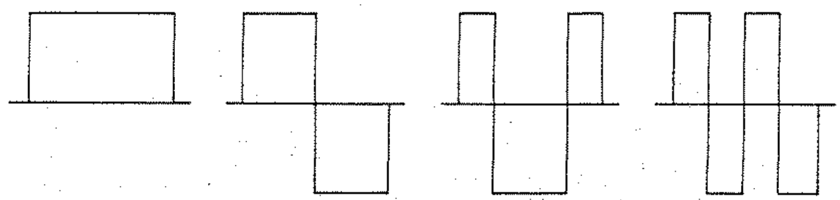


Fig. 4.4. The first four orthogonal Walsh functions.

They look like digital sines and cosines or vectors of 1's and -1's.

(3) We could combine $\sin kx$ with $\sin ly$ into a ***double*** Fourier sine series like

$$f(x, y) = b_{11} \sin x \sin y + b_{21} \sin 2x \sin y + b_{12} \sin x \sin 2y + \cdots. \qquad (25)$$

This applies to a function f that is odd and periodic in both x and y. The interval from $-\pi$ to π has changed into a "period square," and if we know f in one square $-\pi \leq x, y < \pi$ then we know it everywhere. In this case the orthogonal functions are the products $\sin kx \sin ly$. They are orthogonal over the square instead of the interval. They have two indices k and l but they are really just an orthogonal sequence and (25) is an orthogonal expansion.

There is a similar sequence $\cos kx \cos ly$ for even functions (including $k = 0$ and $l = 0$). There will be a larger sequence that mixes sines and cosines, for functions that are neither even or odd. And there will be a corresponding sequence of exponentials—*orthogonal, periodic,* and *complete*:

$$f(x, y) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} c_{kl} e^{ikx} e^{ily}. \qquad (26)$$

These "double Fourier series" can solve Poisson's equation in a square (Ex. 4.1.26).

(4) We could write each of the usual cosines as a polynomial in $\cos \theta$:

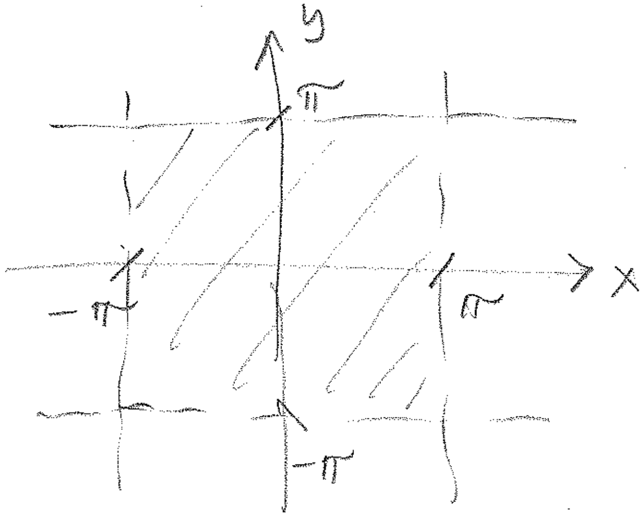
$$\cos 2\theta = 2\cos^2 \theta - 1, \cos 3\theta = 4\cos^3 \theta - 3\cos \theta, \ldots$$

Changing variables to $x = \cos \theta$ gives the ***Chebyshev polynomials*** $T_0 = 1$, $T_1 = x$, $T_2 = 2x^2 - 1$, $T_3 = 4x^3 - 3x$, ... They are orthogonal because the cosines are orthogonal. However there is something new from the change of variables:

$$\int_{-\pi}^{\pi} \cos k\theta \cos l\theta \, d\theta = 0 \text{ becomes } \int_{-1}^{1} T_k(x) T_l(x) \frac{dx}{\sqrt{1 - x^2}} = 0. \qquad (27)$$

Instead of the interval $-\pi \leq x < \pi$ more ... pg. 6

for a 2π-periodic func in one variable, we use a "period square"



We can take the graph on the square & replicate it for periodicity.

There is a similar sequence $\cos kx \cos ly$ for even functions (including $k=0$ & $l=0$).

There is a "larger sequence" that mixes sines and cosines for functions that are neither even nor odd.

There's also a corresponding sequence of orthogonal, periodic, complete exponentials that lets us write

$$F(x,y) = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{kl}\, e^{ikx} e^{ily}$$

These double Fourier series can solve Poisson's eq in a square.

(4) We could write each of the usual cosines as a polynomial in $\cos\theta$ using trig identities

$$\cos 2\theta = 2\cos^2\theta - 1, \quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta, \ldots$$

Changing variables to $x = \cos\theta$ gives the <u>Chebyshev polynomials</u>

$$T_0 = 1, \quad T_1 = x, \quad T_2 = 2x^2 - 1, \quad T_3 = 4x^3 - 3x, \ldots$$

They are orthogonal because cosines are orthogonal.

The change of variables converts   more orthog.

$$\int_{-\pi}^{\pi} \cos k\theta \cos l\theta \, d\theta = 0 \quad \text{to} \quad \int_{-1}^{1} T_k(x) T_l(x) \frac{dx}{\sqrt{1-x^2}} = 0$$

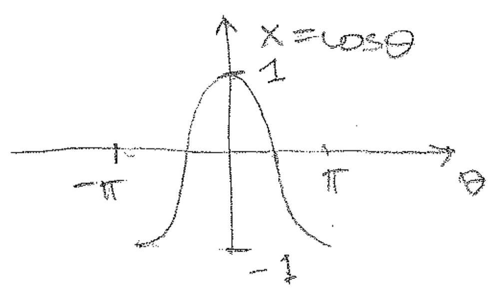
This is weighted orthogonality with weight function $w = \dfrac{1}{\sqrt{1-x^2}}$. It comes from

$x = \cos\theta$

$dx = -\sin\theta \, d\theta \qquad -\sin\theta = -\sqrt{1-\cos^2\theta}$

$dx = -\sqrt{1-x^2} \, d\theta \Rightarrow d\theta = -\dfrac{1}{\sqrt{1-x^2}} dx$

note If $-\pi \le \theta < \pi$, then $x = \cos\theta$ satisfies



$-1 \le x < 1$.