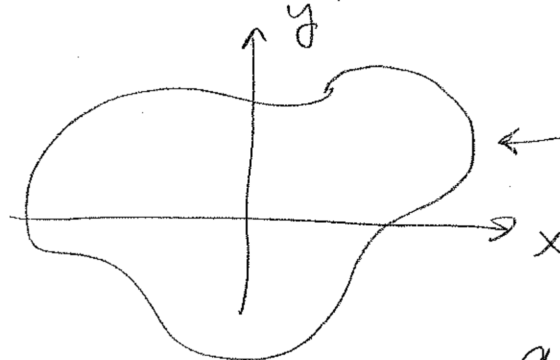
CN 32:

## Application of Fourier Series to Laplace's eq.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Consider Laplaces eq. to hold in a region with some boundary.




If we consider the value of $u$ to be given as $u_0(x,y)$ on the boundary, then this information can be paired with Laplace's eq. to form a boundary value problem (BVP). The unknown fnc. $u(x,y)$ must satisfy Laplace's eq., and $u(x,y)$ must equal $u_0(x,y)$ on the boundary.

specified

Perhaps the BVP models the equilibrium temperature $u(x,y)$ in the region when the temperature is specified to be $u_0(x,y)$ along the boundary.

One solution method is to <u>construct</u> $u$ as an infinite series, choosing the coefficients to match $u_0(x,y)$ on the boundary.

This process depends on the shape of the boundary. Let's consider a disk of radius 1.

In this geometry, the solutions to Laplace's eq. can be written as polynomials in terms of $x$ & $y$, which can be found from

$\mathrm{Re}(x+iy)^n$ and $\mathrm{Im}(x+iy)^n$, $n=0,1,2,\ldots$

notation: $\mathrm{Re}(z)=\mathrm{Re}(x+iy)=x$ : real part of $z$.
$\mathrm{Im}(z)=\mathrm{Im}(x+iy)=y$ : imaginary part of $z$.

$$\mathrm{Re}(x+iy)^0 = \mathrm{Re}(1) = \mathrm{Re}(1+i0) = 1$$

$$\mathrm{Im}(x+iy)^0 = 0$$

$$\mathrm{Re}(x+iy)^1 = x$$

$$\mathrm{Im}(x+iy)^1 = y$$

$$\mathrm{Re}(x+iy)^2 = \mathrm{Re}\left((x+iy)(x+iy)\right) = \mathrm{Re}(x^2-y^2+2ixy)$$
$$= x^2-y^2$$

$$\mathrm{Im}(x+iy)^2 = 2xy$$

the fncs

$u = 0, 1, x, y, x^2-y^2, 2xy, \ldots$ satisfy Laplace's eq $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ bec.

they come from $z = x+iy$: two $y$ derivatives of $(x+iy)^n$ produce a factor of $i^2=-1$, which cancels the effects of two $x$ derivatives.

If you substitute the functions $1, x, y, x^2-y^2, 2xy, \ldots$ into $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ you get a true statement.

Laplace's eq. in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0.$$

You can verify by substitution that $u=r^n\cos n\theta$ & $u=r^n\sin n\theta$ satisfy the eq.

Determining polynomials

$$\mathrm{Re}(z^n) \;\&\; \mathrm{Im}(z^n)$$

is easier in polar coordinates:

$$\begin{aligned}
\mathrm{Re}(z^n) &= \mathrm{Re}\left[(re^{i\theta})^n\right] \\
&= \mathrm{Re}(r^n e^{i\theta n}) \\
&= \mathrm{Re}(r^n(\cos n\theta + i\sin n\theta)) \\
&= \mathrm{Re}(r^n\cos n\theta + i r^n \sin n\theta)
\end{aligned}$$

$$\mathrm{Re}(z^n) = r^n \cos n\theta, \quad n = 0, 1, 2, \ldots$$

$$\mathrm{Im}(z^n) = r^n \sin n\theta, \quad n = 0, 1, 2, \ldots$$

Solutions to Laplace's eq are

$$1, x, y, x^2 - y^2, 2xy, \ldots$$

$$1, r\cos\theta, r\sin\theta, r^2\cos 2\theta, r^2 \sin 2\theta, \ldots$$

combinations of these solutions

give the general solution to Laplace's eq

all sol'ns! $\left( u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos 2\theta + b_2 r^2 \sin 2\theta + \dots \right.$

now we have to choose the constants $a_k$ and $b_k$ to make $u(r,\theta) = u_0(r,\theta)$ on the boundary.

Recall $u_0(r,\theta)$ is taken as a given fnc.

The domain of our problem is a circle of radius 1, so we must have

$$u(1,\theta) = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + \cdots$$

$$= u_0(1,\theta), \quad -\pi < \theta < \pi$$

$\leftarrow$ given

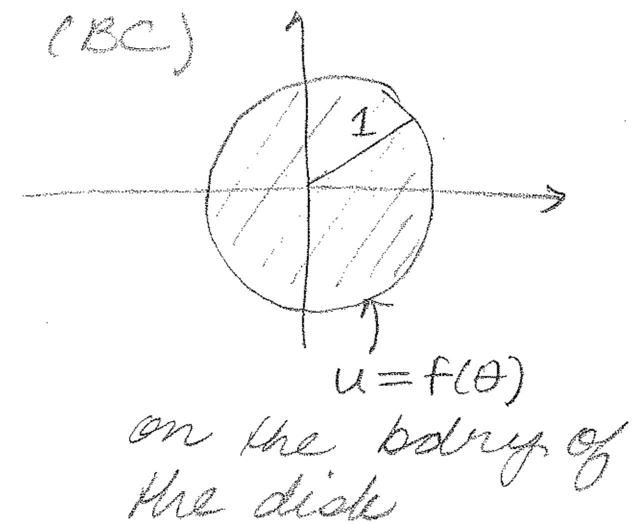
This is the Fourier series for $u_0(1,\theta) = f(\theta)$, which is $2\pi$-periodic bec. $\theta$ & $\theta + 2\pi$ give the same point on the circle.

The constants $a_k$ & $b_k$ must be the Fourier coefficients. Summary:

Laplace's eq. in polar coords on a disk of radius 1: $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$, $-\pi \le \theta \le \pi$, $0 \le r < 1$,

subject the boundary condition (BC)

$$u(1,\theta) = f(\theta)$$




is

$$u(r,\theta) = a_0 + a_1 r\cos\theta + b_1 r\sin\theta + a_2 r^2\cos 2\theta + b_2 r^2 \sin 2\theta + \ldots$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta$.

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(k\theta)\,d\theta, \quad k = 1, 2, \ldots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(k\theta)\,d\theta, \quad k = 1, 2, \ldots$$

FS in LT, 0

The soln to Laplace's eq. on a unit circle with $u = f(\theta)$ on the boundary in exponential form is

$$u(r,\theta) = \sum_{k=-\infty}^{\infty} c_k r^{|k|} e^{ik\theta},$$

where $c_k$ is the Fourier coefficient

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-ik\theta}\, d\theta.$$

Is an infinite series a satisfying soln. to a boundary-value problem?

Yes. We have ways to interpret the series.

use "Fast Fourier transform"

BTW, one can choose the coefficients such that the boundary condition is satisfied at $n$ equally spaced pts around the circle.

Poisson summed the infinite series. His formula gives $u$ directly from the "boundary data" $f(\theta)$ (from the BC $u(1,\theta) = f(\theta)$):

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\,d\phi.$$

note at $r=0$:

$$u(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi.$$

If we interpret $u(r,\theta)$ as the equilibrium temp., this eq says the temp $u$ at the center of the disk (at $r=0$) is the average temp. around the boundary.

★ Poisson's formula gives the temp "response" at $r,\theta$ to the boundary values $f(\theta)$.

We can verify Poisson's formula satisfies Laplace's eq. by substituting

$$\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} \quad \text{into} \quad \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

and getting 0.