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18%. This is the ***Gibbs***
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CN 30:

Sine Series and Cosine Series

The first two examples produced only cosines; the third function $f(x)=x$ produced only sines. There will be no mystery if we look again at even and odd functions:

$$f \text{ is } \textbf{\textit{even}} \text{ if } \quad f(-x)=f(x) \qquad \text{for all } x$$

$$f \text{ is } \textbf{\textit{odd}} \text{ if } \quad f(-x)=-f(x) \quad \text{for all } x$$

In either case it is enough to know f over the *half*-period $0 \le x < \pi$, since in the other half its graph is either a mirror image (the even case) or its negative.

Cosines are even and sines are odd. The function x^k is even if k is even, and odd if k is odd. Exponentials e^{ikx} and most other functions are neither. But every function has an even part and an odd part,

$$f=f_e+f_o, \quad \text{with} \quad f_e(x)=\frac{f(x)+f(-x)}{2} \quad \text{and} \quad f_o(x)=\frac{f(x)-f(-x)}{2}$$

The even part yields the cosine terms and the odd part yields the sines.

4B If f is odd then all a_k are zero and all b_k can be computed from the half-period between 0 and π:

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx=\frac{2}{\pi}\int_0^{\pi} f(x)\sin kx\,dx. \tag{13}$$

Similarly b_k is zero if f is even, and a_k comes from a half-period.

That idea can be turned around. Suppose f is known over the half-period $0<x<\pi$. Then the b_k from (13) are the coefficients in its ***Fourier sine series***

$$f(x)=b_1\sin x+b_2\sin 2x+b_3\sin 3x+\cdots \tag{14}$$

This series will produce f on the other half-period, where it was originally not known, provided f is extended in the right way: Since the sines are odd, f should extend past $x=0$ as an odd function. Then the Fourier series for f has no cosine terms and (14) is correct over the whole period.

That is an odd periodic extension of f. The opposite case keeps only the cosine coefficients

$$a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\,dx \quad \text{and} \quad a_k=\frac{2}{\pi}\int_0^{\pi} f(x)\cos kx\,dx.$$

These give the ***Fourier cosine series***, and it matches f over a whole period provided f is extended to be even: $f(-x)=f(x)$.

Introduction to applied mathematics, Strang