CN 29 Fourier Series Examples

Determine a F.S. expansion for $f(x)$.

(1) $f(x) = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$ (trig identity), $-\pi < x < \pi$

$= a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$,

where $a_0 = \frac{1}{2}$, $a_1 = 0$, $b_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = a_4 = a_5 = \cdots = 0$

$b_2 = b_3 = b_4 = \cdots = 0$.

(2) $f(x) = x$

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx,$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx, \quad k = 1, 2, 3, \ldots$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\left.\frac{x^2}{2}\right|_{-\pi}^{\pi} = \frac{1}{4\pi}\left[\pi^2 - (-\pi)^2\right]$$

$$= \frac{1}{4\pi}\left[\pi^2 - \pi^2\right] = 0.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{\text{odd}}{x}\, \underbrace{\cos kx}_{\text{even}}\,dx = 0$$

integrand is an odd fnc. ~~Interval~~ $(-\pi, \pi)$ is symmetrical about 0.

(2) continued

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x \sin kx}_{\text{odd} \cdot \text{odd}}\, dx$$

Integrand is an even func. Interval $(-\pi, \pi)$ is symmetrical about 0.

$$= \frac{2}{\pi}\int_0^{\pi} x \sin kx \, dx$$

$u = x \qquad dv = \sin kx\, dx$

$du = dx \qquad v = -\frac{\cos kx}{k}$

$$= \frac{2}{\pi}\left[uv\Big|_0^{\pi} - \int_0^{\pi} v\, du \right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{k}\, x \cos kx \Big|_0^{\pi} - \int_0^{\pi} -\frac{1}{k} \cos kx \, dx \right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{k}\, x \cos kx + \frac{1}{k}\,\frac{\sin kx}{k} \right]\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi k}\left[-x \cos kx + \frac{1}{k} \sin kx \right]_0^{\pi}$$

$$= \frac{2}{\pi k}\left[-\pi \cos k\pi + 0 - (-0 + 0) \right] = -\frac{2}{k} \cos k\pi$$

$$b_k = -\frac{2}{k}\cos k\pi = \begin{cases} -\frac{2}{k}(1) \text{ if } k \text{ even} \\ -\frac{2}{k}(-1) \text{ if } k \text{ odd.} \end{cases}$$

$$= -\frac{2}{k}(-1)^k \quad , \; k = 1, 2, 3, \ldots$$

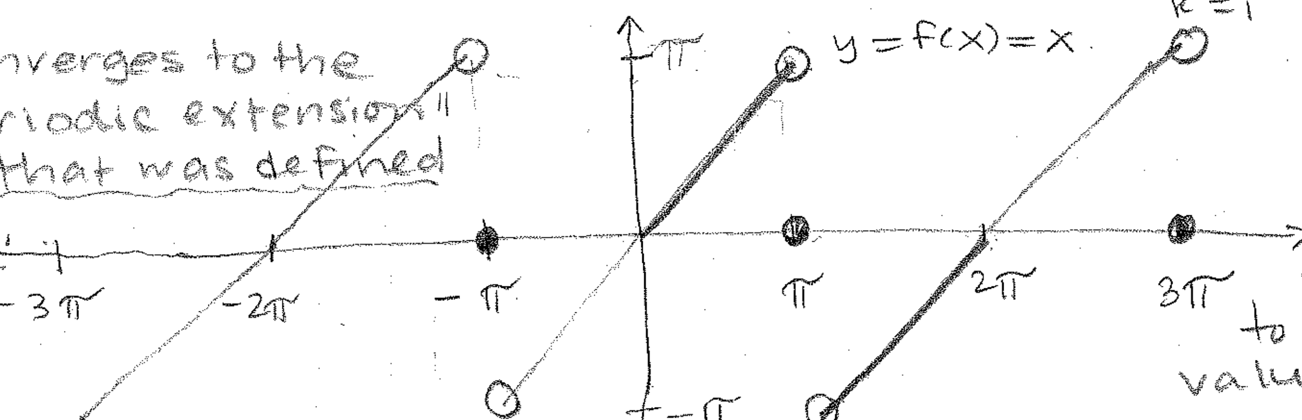
$$\overset{0}{a_0} + \overset{0}{a_1}\cos x + b_1 \sin x + \overset{0}{a_2}\cos 2x + b_2 \sin 2x + \overset{0}{a_3}\cos 3x + b_3 \sin 3x + \cdots$$

$$= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

$$= \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x + \cdots$$

$$= 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right) = \sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \sin kx$$

F.S. converges to the "2π-periodic extension" of $f(x)$ that was defined on $(-\pi, \pi)$.



Expect a 2π-periodic fnc.

The F.S. converges to the average value of the jumps.

FS exs, 5

there are two things to notice:

(1) The series gives the answer zero at $x = \pi$. This answer is halfway between the values of f on opposite sides of the discontinuity.

(2) Near the jump there is an overshoot of about 18%. This is the ***Gibbs phenomenon***, which does not disappear as more Fourier terms are kept—it moves closer to the jump.

Both 1 and 2 are typical of Fourier series at a discontinuity. Since the step function at a jump is the integral of a delta function, the overshoot and oscillations in Fig. 4.3 must be the integral of Fig. 4.2. The area under the central arch in Fig. 4.2 is nearly 1.18.†

Another typical and important feature of jumps is the slow decay of the Fourier coefficients. As in (12), the kth coefficient is of order $1/k$; that reappears for the square wave in the exercises. For its integral the hat function, a_k and b_k decay like $1/k^2$. For its derivative the delta function, a_k and b_k are constant.

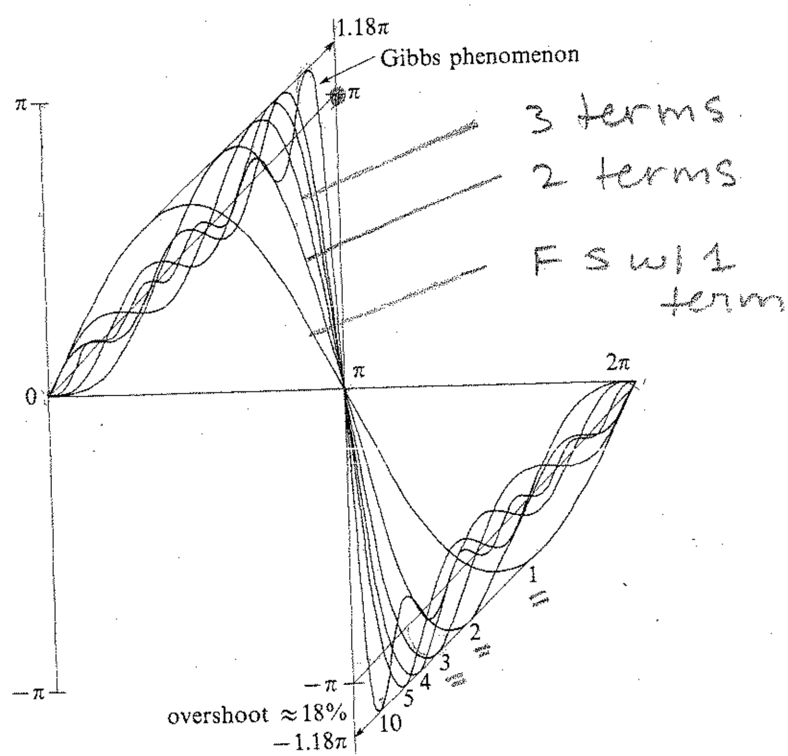


Fig. 4.3. Partial sums of the Fourier series for $f = x$.

† This number is not the same in all books.