N26: <u>Standard inner product on a vector space of functions</u>

If the set S is a vector space of real-valued functions over $\mathbb{R}$, where the functions f are defined on $a_1 < x < a_2$, then a standard inner inner product is:

$$\langle f, g \rangle = \int_{a_1}^{a_2} f(x)\, g(x)\, dx$$

a function that assigns to each ordered pair of elements $f, g \in S$ a scalar $\langle f, g \rangle$.

For $f, g, h \in S$ and $\alpha \in \mathbb{R}$, check properties 1–4 hold, noting f, g, h are functions defined on $a_1 < x < a_2$.

① $\langle f+g, h \rangle \stackrel{?}{=} \langle f, h \rangle + \langle g, h \rangle$

$$\langle f+g, h \rangle = \int_{a_1}^{a_2} (f+g)(x)\, h(x)\, dx = \int_{a_1}^{a_2} (f(x)+g(x))\, h(x)\, dx$$

$$= \int_{a_1}^{a_2} (f(x)\, h(x) + g(x)\, h(x))\, dx = \int_{a_1}^{a_2} f(x) h(x) dx + \int_{a_1}^{a_2} g(x) h(x) dx$$

$$= \langle f, h \rangle + \langle g, h \rangle . \checkmark$$

② $\langle \alpha f, g \rangle \stackrel{?}{=} \alpha \langle f, g \rangle$

$$\langle \alpha f, g \rangle = \int_{a_1}^{a_2} (\alpha f)(x)\, g(x)\, dx = \int_{a_1}^{a_2} \alpha f(x) g(x)\, dx$$

$$= \alpha \int_{a_1}^{a_2} f(x) g(x)\, dx = \alpha \langle f, g \rangle \checkmark$$

③ $\langle g, f \rangle \stackrel{?}{=} \overline{\langle f, g \rangle}$

$$\langle g, f \rangle = \int_{a_1}^{a_2} g(x) f(x)\, dx$$

$$= \int_{a_1}^{a_2} f(x) g(x)\, dx$$

$$= \overline{\int_{a_1}^{a_2} f(x) g(x)\, dx}$$

because $\int_{a_1}^{a_2} f(x) g(x)\, dx$ is real

$$= \overline{\langle f, g \rangle} \checkmark$$

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④ $\langle f, f \rangle \overset{?}{>} 0$ if $f \neq 0$

$$\langle f, f \rangle = \int_{a_1}^{a_2} [f(x)]^2 \, dx > 0$$ if f is not the zero function.

"area under a curve" that lies above the x-axis is positive.

Standard inner product on a vector space of complex-valued functions: $\mathbb{R} \to \mathbb{C}$

If the set S is a vector space of complex-valued functions: $\mathbb{R} \to \mathbb{C}$ over $\mathbb{C}$, where the functions f are defined on $a_1 < x < a_2$, then a standard (complex-valued) inner product is

$$\langle f, g \rangle = \int_{a_1}^{a_2} f(x)\overline{g(x)}\,dx$$

← a fnc that assigns to each ordered pair of elements $f, g \in S$ a scalar $\langle f, g \rangle$.

For $f, g, h \in S$ and $\alpha \in \mathbb{C}$, check Properties 1–4 of an inner product hold, noting f, g, h are functions defined on $a_1 < x < a_2$.

① $\langle f+g, h \rangle \stackrel{?}{=} \langle f, h \rangle + \langle g, h \rangle$

$$\langle f+g, h \rangle = \int_{a_1}^{a_2} (f+g)(x)\overline{h(x)}\,dx = \int_{a_1}^{a_2} (f(x)+g(x))\overline{h(x)}\,dx$$

$$= \int_{a_1}^{a_2} f(x)\overline{h(x)}\,dx + \int_{a_1}^{a_2} g(x)\overline{h(x)}\,dx = \langle f, h \rangle + \langle g, h \rangle$$

✓

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② $\langle \alpha f, g \rangle \overset{?}{=} \alpha \langle f, g \rangle$

$$\langle \alpha f, g \rangle = \int_{a_1}^{a_2} (\alpha f)(x)\overline{g(x)}\,dx = \int_{a_1}^{a_2} \alpha f(x)\overline{g(x)}\,dx$$

$$= \alpha \int_{a_1}^{a_2} f(x)\overline{g(x)}\,dx = \alpha \langle f, g \rangle \checkmark$$

③ $\langle g, f \rangle \overset{?}{=} \overline{\langle f, g \rangle}$

$$\langle g, f \rangle = \int_{a_1}^{a_2} g(x)\overline{f(x)}\,dx$$

$$\overline{\langle f, g \rangle} = \overline{\int_{a_1}^{a_2} f(x)\overline{g(x)}\,dx}$$

are these equal?

$$\overline{\int_{a_1}^{a_2} f(x)\overline{g(x)}\,dx} \overset{?}{=} \int_{a_1}^{a_2} g(x)\overline{f(x)}\,dx.$$

Let $f(x) = a(x) + i\,b(x)$, $g(x) = c(x) + i\,d(x)$.

$$\int_{a_1}^{a_2} f(x)\,\overline{g(x)}\,dx = \int_{a_1}^{a_2} [a(x)+ib(x)][c(x)-id(x)]\,dx$$

$$= \int_{a_1}^{a_2} [a(x)c(x) - ia(x)d(x) + ib(x)c(x) + b(x)d(x)]\,dx$$

$$= \int_{a_1}^{a_2} [a(x)c(x) + b(x)d(x) + i\{-a(x)d(x) + b(x)c(x)\}]\,dx$$

$$= \int_{a_1}^{a_2} [a(x)c(x) + b(x)d(x)]\,dx + i\int_{a_1}^{a_2} [-a(x)d(x) + b(x)c(x)]\,dx$$

$$= \int_{a_1}^{a_2} [a(x)c(x) + b(x)d(x)]\,dx - i\int_{a_1}^{a_2} [-a(x)d(x) + b(x)c(x)]\,dx$$

$$\int_{a_1}^{a_2} g(x)\,\overline{f(x)}\,dx = \int_{a_1}^{a_2} [c(x)+id(x)][a(x)-ib(x)]\,dx$$

$$= \int_{a_1}^{a_2} [a(x)c(x) + b(x)d(x) + i\{a(x)d(x) - b(x)c(x)\}]\,dx$$

$$= \int_{a_1}^{a_2} [a(x)c(x) + b(x)d(x)]\,dx - i\int_{a_1}^{a_2} [-a(x)d(x) + b(x)c(x)]\,dx$$

=

(4) $\langle f, f \rangle \overset{?}{>} 0$ if $f \neq 0$

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$$\langle f, f \rangle = \int_{a_1}^{a_2} f(x)\overline{f(x)}\, dx = \int_{a_1}^{a_2} [a(x)+ib(x)][a(x)-ib(x)]\,dx$$

$$= \int_{a_1}^{a_2} \{[a(x)]^2 + [b(x)]^2\}\, dx > 0 \text{ if}$$

$f(x)$ is not the zero function.

"area under a curve" that lies above the x-axis is positive.