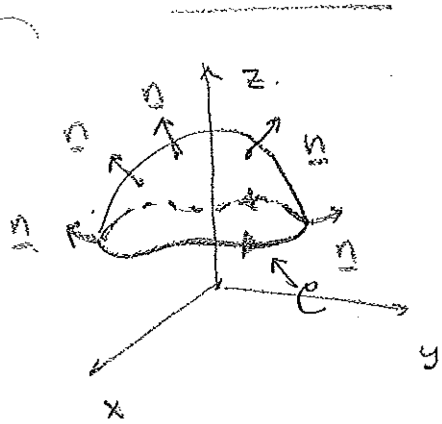
CN 24:
Stokes' thm.



- let S be piecewise smooth & oriented.
- surface S has a boundary curve C. The orientation of S (via $\underline{n}$) induces a positive orientation on C.

- Positive means if you walk along C w/ the top of your head pointing in the direction $\underline{n}$, then the surface is on your left.

- Notation: $C = \partial S$ means "boundary of S."
- let ∂S be simple (not crossing itself) & closed.
- let vector field $\underline{F}$ have components w/ continuous 1st partial derivatives on an open (*) region in $\mathbb{R}^3$ that contains S.

☆ Then $$\int_{\partial S} \underline{F} \cdot d\underline{r} = \iint_{S} \text{curl}\,\underline{F} \cdot d\underline{S}.$$

The line integral of the tangential comp't of $\underline{F}$ around the bdry curve ∂S = surface int. of the normal comp't of curl $\underline{F}$.

[(*) Recall an open region does not contain its bdry points, e.g. the set of points inside but not on a sphere.]

- The thm is useful if one side of eq. is easier than other.
- Notice Stokes' Thm says flux integrals of $\nabla \times \underline{F}$ over two different oriented surfaces are the same if the surfaces have the same oriented boundary curve. So the Thm. is also useful if one such surface is simpler than the other.

Stokes' Thm, 3

Ex. 1. Use Stokes' Thm to eval $\iint_S \text{curl}\,\underline{F}\cdot d\underline{S} = I$

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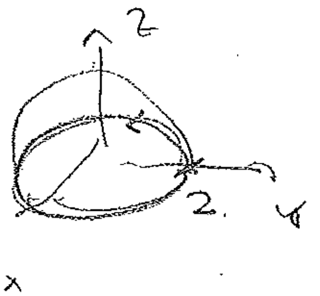
where

$$\underline{F} = x^2 e^{yz}\,\underline{i} + y^2 e^{xz}\,\underline{j} + z^2 e^{xy}\,\underline{k}$$

& S is the hemisphere

$$S: x^2+y^2+z^2 = 4, \quad z \geq 0. \quad \text{oriented up.}$$



$$I = \int_{\partial S} \underline{F}\cdot d\underline{r}$$

$$\underline{r}(t) = 2\cos t\,\underline{i} + 2\sin t\,\underline{j} + 0\,\underline{k}$$
$$\underline{r}'(t) = -2\sin t\,\underline{i} + 2\cos t\,\underline{j} + 0\,\underline{k}.$$

$$I = \int_{\partial S} \underline{F}(t)\cdot \frac{d\underline{r}}{dt}\,dt =$$

$$= \int_0^{2\pi} \langle 4\cos^2 t, 4\sin^2 t, 0\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\,dt$$

$$= \int_0^{2\pi} \left(-8\sin t\cos^2 t + 8\cos t\sin^2 t\right)dt =$$

$$= 8\left(\frac{\cos^3 t}{3} + \frac{\sin^3 t}{3}\right)\Big|_0^{2\pi} = \frac{8}{3}[1-1] = 0. \checkmark$$

Interpret w/ MVT:

$$\underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2e^{yz} & y^2e^{xz} & z^2e^{xy} \end{vmatrix} = \underline{i}\,(z^2e^{xy}x - y^2e^{xz}x) - \underline{j}\,(z^2e^{xy}y - x^2e^{yz}y) + \underline{k}\,(y^2e^{xz}z - x^2e^{yz}z)$$

swirling around inside but not thru sphere: No flux across S.