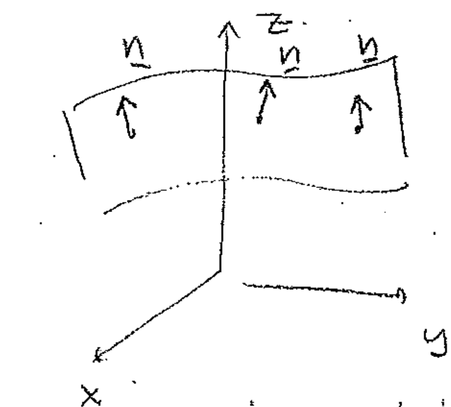
CN22 Surface Integrals of vector Fields.

oriented surfaces.

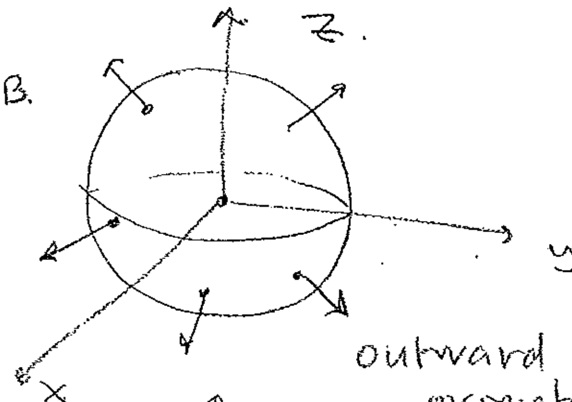
"def": An oriented surface S is two-sided.

A



upward orientation.

or B.



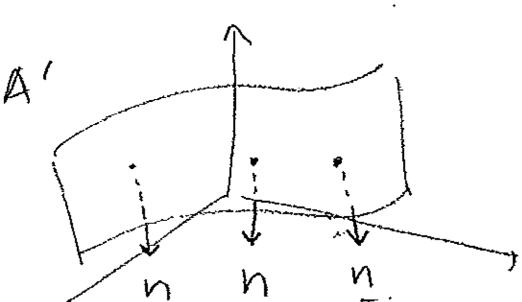
outward orientation

closed surface.

It is oriented by the unit normal vector $\underline{n}$ to S.

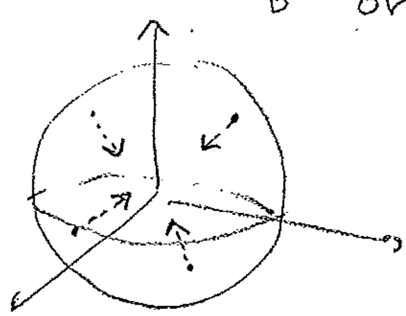
Two choices. A

OR A'



downward orientation.

B or B'



inward orientation

But $\underline{n}$ doesn't jump from upward to downward.

Cen

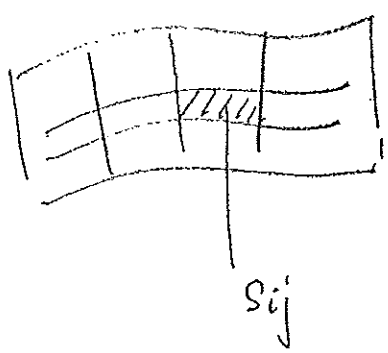
"Def" If S is smooth, it has a unique tangent plane at each pt (x,y,z) on S. (no cusps).

Suppose a fluid w/ density $\rho(x,y,z)$ & velocity field $\underline{v}(x,y,z)$ flows across S.

(& S doesn't impede flow).

Then $(\rho\,\underline{v})\cdot\underline{n}$ is mass that passes though unit area per unit time. in the direction of $\underline{n}$.

$\uparrow$ $\frac{\text{mass}}{\cancel{\text{vol}}\ \text{area}}$ $\nwarrow$ $\frac{\cancel{\text{length}}}{\text{time}}$



$(\rho\underline{v}\cdot\underline{n})\,A(S_{ij})$ is mass flowing orthogonally through S_{ij} per unit time.

In the limit, get $\iint_S \rho\underline{v}\cdot\underline{n}\,dS =$

$$\iint_S \rho(x,y,z)\,\underline{v}(x,y,z)\cdot n(x,y,z)\,dS$$

$=$ mass that flows across S per unit time.

$\underline{F} = \rho \underline{v} \Rightarrow \iint_S \underline{F} \cdot \underline{n} \, dS.$

$\underbrace{\text{surface integral}}$ of $\underline{F}$ over S

(flux integral).

Notation : $\boxed{\iint_S \underline{F} \cdot d\underline{S} = \iint_S \underline{F} \cdot \underline{n} \, dS.}$

flux of $\underline{F}$ across S.

Given S, how do we calculate $\underline{n}$?

Lea

Surface S could be defined via

$$z = g(x,y), \quad x = h_1(y,z), \quad y = h_2(x,z).$$

or $\underbrace{F(x,y,z) = 0}_{(*)}$ ⟵ Here $\underline{\nabla} F \perp S$.

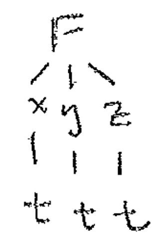
Why? $\left\{ \begin{array}{l} \text{Let any curve } C \text{ on } S \\ \text{be parametrized by} \end{array} \right.$

$$\underline{r} = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$$

Diff. (*) w.r.t. t:

$$\frac{dF}{dt} = 0$$

$$\Longleftrightarrow \quad \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$



$$\Longleftrightarrow \quad \underline{\nabla} F \cdot r'(t) = 0$$

$$\underline{\nabla} F \perp \underline{r}'(t)$$

This is tangent to C & therefore to S.

This is orthogonal to the tangent to C for every C on the surface we choose, so it is therefore normal to S.

If S is defined by $z = g(x,y)$, let $z - g(x,y) = F(x,y,z) = 0$

$$\underline{n} = \frac{\underline{\nabla} F}{\|\nabla F\|} = \frac{-\frac{\partial g}{\partial x}\underline{i} - \frac{\partial g}{\partial y}\underline{j} + \underline{k}}{\sqrt{(g_x)^2 + (g_y)^2 + 1}}$$

(upward orientation)

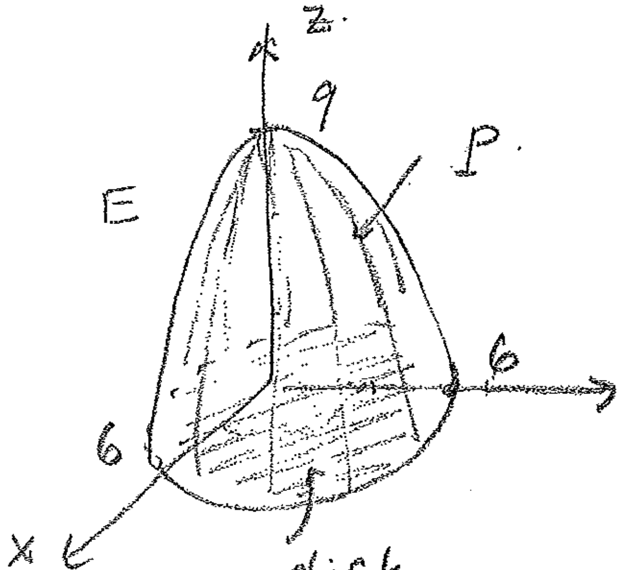
Note the denominator will cancel

in $\iint_S \underline{F} \cdot \underline{n} \, dS$:

$$\iint_D \underline{F} \cdot \frac{(-g_x \underline{i} - g_y \underline{j} + \underline{k})}{\cancel{\sqrt{(g_x)^2+(g_y)^2+1}}} \cancel{\sqrt{(g_x)^2+(g_y)^2+1}} \, dA$$

Prob. a ~~fluid~~ with density 1200 units of mass per volume flows with velocity $\underline{V} = y\,\underline{i} + \underline{j} + z\,\underline{k}$. Find the flow rate outward through the boundary surface of the solid region E bounded by the paraboloid $z = 9 - \frac{1}{4}(x^2+y^2)$ & the disk $x^2+y^2 \le 36$ and the xy-plane ($z=0$ plane) by (a) calculating the surface integral of a vector field and (b) applying the divergence theorem.

Go



(a) $\iint_S \rho \underline{v} \cdot \underline{n}\, dS = I$

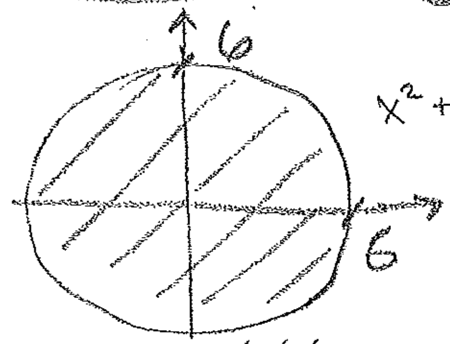
$= \underbrace{\iint_P \rho \underline{v} \cdot \underline{n}\, dS}_{①} + \underbrace{\iint_{disk} \rho \underline{v} \cdot \underline{n}\, dS}_{②}$

where does P intersect xy-plane?

$z = 9 - \frac{1}{4}(x^2+y^2) = 0$

$9 = \frac{1}{4}(x^2+y^2)$

$x^2+y^2 = 36$



$x^2+y^2 \le 36$

① $\iint_P \rho \underline{v} \cdot \underline{n}\, dS = 1200 \iint_D (y\underline{i} + \underline{j} + z\underline{k}) \cdot \left(-\frac{\partial g}{\partial x}\underline{i} - \frac{\partial g}{\partial y}\underline{j} + 1\underline{k}\right) dA$ $\quad z = 9 - \frac{1}{4}(x^2+y^2) = g(x,y)$

$-\frac{\partial g}{\partial x} = \frac{1}{2}x, \quad -\frac{\partial g}{\partial y} = \frac{1}{2}y$

$1200 \iint_D \left(\frac{1}{2}xy + \frac{1}{2}y + \left(9 - \frac{1}{4}(x^2+y^2)\right)\right) dA$ (*)

$= \cdots = 194{,}400\pi$ mass/time

② $\iint_{disk} \rho \underline{v} \cdot \underline{n}\, dS = 1200 \iint_D (y\underline{i} + \underline{j} + 0\underline{k}) \cdot -\underline{k}\, dA$

$= 0.$

$$I = \frac{1200}{2}\int_0^{2\pi}\int_0^6 \left[\tfrac{1}{2}r^3\cos\theta\sin\theta + \tfrac{1}{2}r^2\sin\theta + 2(9r - \tfrac{1}{4}r^3)\right] dr\,d\theta$$

$$= \frac{1200}{2}\int_0^6 \left[\frac{\sin^2\theta}{2}r^3 - r^2\cos\theta + 2(9r - \tfrac{1}{4}r^3)\theta\right]_0^{2\pi}$$

$$= 2(600)\int_0^6 (9r - \tfrac{1}{4}r^3)\cdot 2\pi\,dr.$$

Preview: Divergence thm takes you quickly to here.

$$= 2400\pi\left(\frac{9r^2}{2} - \frac{1}{4}\frac{r^4}{4}\right)\Bigg|_0^6$$

$$= 2400\pi\left(9\cdot 18 - \frac{6\cdot 6\cdot 6\cdot 6}{16}\right)$$

$$= 2400\pi\ 3^4$$

$$= 2400\pi(81)$$

$$= 194{,}400\cdot\pi \ \ \text{units mass / time}$$