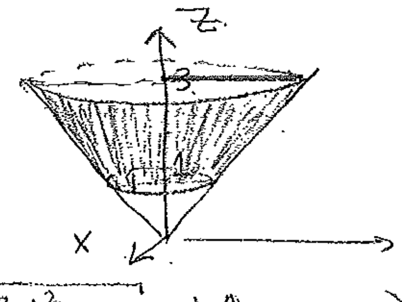
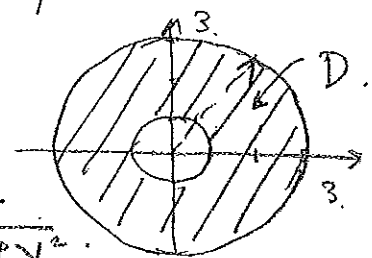
CN21 Surface Integration Problems

Ex. Find the mass of a funnel in the shape of the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ & $z = 3$ if its density is $\rho(x,y,z) = x^2 z^2$.



$$M = \iint_S x^2 z^2 \, dS$$

$$= \iint_D \rho(x, y, g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \; dA$$



$$z = g(x,y) = \sqrt{x^2 + y^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{\cancel{2}} \frac{\cancel{2}x}{\sqrt{x^2+y^2}}, \quad \frac{\partial z}{\partial y} = \frac{1}{\cancel{2}} \frac{\cancel{2}y}{\sqrt{x^2+y^2}}$$

$$\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{\frac{x^2 + y^2 + (x^2+y^2)}{x^2+y^2}} = \sqrt{2}.$$

$$M = \iint_D \underbrace{x^2}\,(x^2+y^2)\sqrt{2}\; \underline{\underline{dA}}$$

D is a polar type region.

$x = r\cos\theta$
$y = r\sin\theta$

$$= \int_0^{2\pi}\int_1^3 \underbrace{r^2\cos^2\theta}\, r^2 \sqrt{2}\; \underline{\underline{r\,dr\,d\theta}}$$

$$= \sqrt{2} \int_0^{2\pi} \cos^2\theta \left(\int_1^3 r^5\,dr\right) d\theta$$

$$= \sqrt{2}\left(\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\,d\theta\right)\left(\frac{r^6}{6}\Big|_1^3\right)$$

$$= \sqrt{2}\left(\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi} \left[\frac{3^6}{6} - \frac{1}{6}\right]$$

$$= \sqrt{2}\,\pi\left(\frac{728}{6}\right) = \frac{364}{3}\sqrt{2}\,\pi \text{ units of mass.}$$

(cont.)

Ex. Find mass of surface S given by

$x = y + 2z^2$, $0 \le y \le 1$.

$0 \le z \le 1$. w/ density $\rho(x,y,z) = z$.

$$M = \iint_S z \, dS$$

$$\iint_D \rho(h(y,z), y, z) \sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1} \; dA$$

$$\int_0^1 \int_0^1 z \cdot \sqrt{1^2 + (4z)^2 + 1} \; dy \cdot dz$$

$$= \int_0^1 \int_0^1 z \sqrt{16z^2 + 2} \; dy \, dz$$

$$= \int_0^1 z \sqrt{16z^2 + 2} \; (y)\Big|_0^1 \; dz$$

$$= \frac{\cancel{z}}{3} \frac{(16z^2 + 2)^{3/2}}{\cancel{3/2} \cdot 16} \Bigg|_0^1$$

$$= \frac{1}{48} \left((\sqrt{18})^3 - (\sqrt{2})^3 \right)$$

$\sqrt{18} = \sqrt{9 \cdot 2}$

$= 3\sqrt{2}$

$$= \frac{\cancel{2}}{48} \left[27 \cdot \cancel{2}\sqrt{2} - \cancel{2}\sqrt{2} \right)$$

$$= \frac{1}{\cancel{24}_{12}} (\cancel{26}^{13} \sqrt{2})$$

$$= \frac{13}{12}\sqrt{2} \text{ units mass.}$$