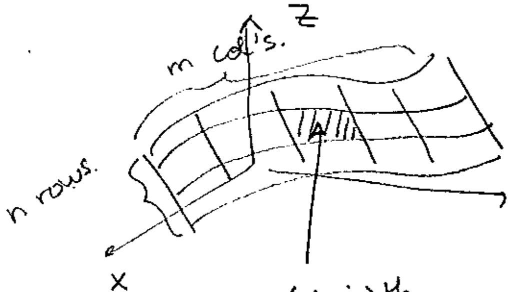
CN 18: Surface Integrals. Dr. Laura Gross

Let $f = f(x,y,z)$ be scalar fnc. whose domain includes a surface S.

Divide S into patches S_{ij} w/ area ΔS_{ij}.

e.g. $[f] = \frac{\text{mass}}{\text{area}}$



← domain of f from $\lim \frac{\Delta m_{ij}}{\Delta S_{ij}}$

$(i,j)^{th}$ patch. S_{ij} contains pt P_{ij}^* & has area ΔS_{ij}

def $$\lim_{m,n\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} \underbrace{\underbrace{f(P_{ij}^*)}_{\substack{\text{density}\\\text{of patch}\\\text{almost}\\\text{constant}}} \Delta S_{ij}}_{\substack{\text{aproximate}\\\text{mass of}\\\text{patch}}} = \underbrace{\iint_S f(x,y,z)\, dS}_{\substack{\text{surface int of}\\ f(x,y,z) \text{ on surface } S}}$$

- If $f(x,y,z)$ is a density function $\rho(x,y,z)$ in units of mass per area, then the surface integral gives the mass M of the sheet depicted in the figure.

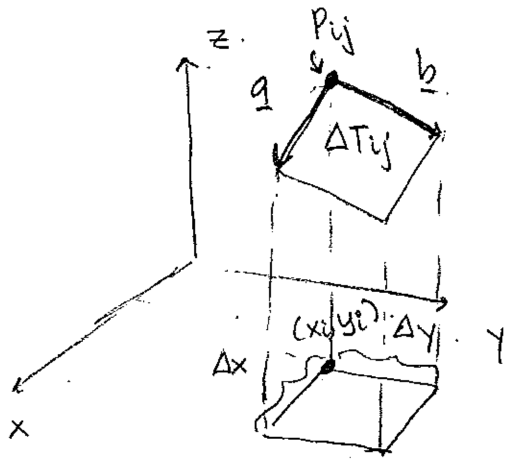
- The center of mass is $(\bar{x}, \bar{y}, \bar{z}) =$
$$\left(\frac{1}{M}\iint_S x\rho(x,y,z)\, dS,\ \frac{1}{M}\iint_S y\rho\, dS,\ \frac{1}{M}\iint_S z\rho\, dS\right).$$

- Notice surface area $= A(S) = \iint_S dS$.

Suppose surface S is def^{d} by

$$z = g(x,y).$$

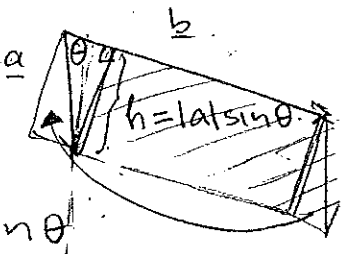
We can aproximate area ΔS_{ij} w/ area of the tangent plane ΔT_{ij} to S_{ij} at the pt P_{ij}^{*}.



$$\underline{a} = \Delta x\,\underline{i} + 0\,\underline{j} + m_{(x)}\Delta x\,\underline{k}$$
$$= \Delta x\,\underline{i} + \frac{\partial g}{\partial x}(x_i, y_i)\,\Delta x\cdot\underline{k}$$

$$\underline{b} = 0\,\underline{i} + \Delta y\,\underline{j} + m_{(y)}\Delta y\,\underline{k}$$
$$= \Delta y\,\underline{j} + \frac{\partial g}{\partial y}(x_i, y_i)\,\Delta y\,\underline{k}.$$

$$\Delta T_{ij} = \|\underline{a}\times\underline{b}\|$$



$$\text{bec. } \|\underline{a}\times\underline{b}\| = \begin{cases} \|\underline{b}\|\,h \\ \|\underline{a}\|\,\|\underline{b}\|\sin\theta \end{cases}$$

$$\sin\theta = \frac{h}{\|a\|}.$$

$$= \left\| \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \Delta x & 0 & g_x(x_i,y_i)\Delta x \\ 0 & \Delta y & g_y(x_i,y_i)\Delta y \end{vmatrix} \right\|$$

$$= \| \underline{i}\,(-g_x(x_i,y_i)\Delta y\cancel{\Delta x}) - \underline{j}\,(\Delta x\, g_y(x_i,y_i)\Delta y) + \underline{k}\,\Delta x\Delta y \|$$

$$\Delta T_{ij} = \sqrt{g_x^2(x_i,y_i) + g_y^2(x_i,y_i) + 1}\;\Delta x\,\Delta y$$

area of projection of ΔT_{ij} onto xy-plane

$$dS = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\;dA.$$

Surface integrals

① surface $z = g(x,y)$.

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}\,dA$$

where D is the projection of S onto xy-plane.

② analogously.... for surface $y = h_1(x,z)$.

$$\iint_S f(x,y,z)\,dS = \iint_{D_1} f(x,h_1(x,z),z)\sqrt{\left(\frac{\partial h_1}{\partial x}\right)^2+\left(\frac{\partial h_1}{\partial z}\right)^2+1}\,dA$$

where D_1 is the projection of S onto xz-plane

③ or.... for surface $x = h_2(y,z)$.

$$\iint_S f(x,y,z)\,dS = \iint_{D_2} f(h_2(y,z),y,z)\sqrt{\left(\frac{\partial h_2}{\partial y}\right)^2+\left(\frac{\partial h_2}{\partial z}\right)^2+1}\,dA$$

where D_2 is the projection of S onto yz-plane

Alternate notation: $\left\{\begin{array}{l} z = g(x,y) \rightarrow \\ \dfrac{\partial g}{\partial x} = \dfrac{\partial z}{\partial x}, \quad \dfrac{\partial g}{\partial y} = \dfrac{\partial z}{\partial y} \end{array}\right.$

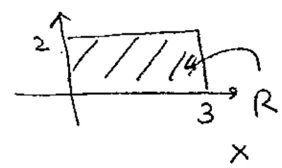
$\left\{\begin{array}{l} y = h_1(x,z) \rightarrow \\ \dfrac{\partial h_1}{\partial x} = \dfrac{\partial y}{\partial x}, \quad \dfrac{\partial h_1}{\partial z} = \dfrac{\partial y}{\partial z} \end{array}\right.$

$\left\{\begin{array}{l} x = h_2(y,z) \rightarrow \\ \dfrac{\partial h_2}{\partial y} = \dfrac{\partial x}{\partial y}, \quad \dfrac{\partial h_2}{\partial z} = \dfrac{\partial x}{\partial z} \end{array}\right.$

Ex1 evaluate $I = \iint_S x^2 y z \, dS$, where S is given by $z = \underbrace{1+2x+3y}_{= g(x,y)}$ above rectangle $[0,3]\times[0,2]$

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{1+2^2+3^2} = \sqrt{14}.$$

$$I = \iint_R x^2 y (1+2x+3y)\sqrt{14}\, dA$$



$$= \sqrt{14}\int_0^2 \int_0^3 \left(x^2 y + 2x^3 y + 3x^2 y^2\right) dx\, dy$$

$$= \sqrt{14}\int_0^2 \left[\frac{y x^3}{3} + \frac{x^4 y}{2} + x^3 y^2 \right]_{x=0}^{3} dy$$

$$= \sqrt{14}\int_0^2 \left(y\,9 + \frac{y \cdot 3^4}{2} + 27 y^2 \right) dy$$

$$= \sqrt{14}\left[\frac{9y^2}{2} + \frac{3^4 y^2}{4} + 9y^3 \right]_0^2$$

$$= \sqrt{14}\left(18 + 3^4 + 72\right)$$

$$= 171 \sqrt{14} \quad \checkmark$$

81
72
18
171

e.g. $[I]$ = mass
if $[x^2yz]$ are $\frac{\text{mass}}{\text{area}}$.
I = mass of the portion of the plane $z = 1+2x+3y$