CN 15: Conservative Vector Fields

Gross

Fundametal Theorem of Line integration.

Q: why does $\int_C \underline{F} \cdot d\underline{r} = \varphi(\underline{r}(b)) - \varphi(\underline{r}(a))$

for conservative vector fields?

$$\int_C \underline{F} \cdot d\underline{r} = \int_C \underline{\nabla} \varphi \cdot d\underline{r} \quad (\text{if } \underline{F}(\underline{r}) = \underline{\nabla}\varphi(\underline{r}))$$

$$= \int_a^b \nabla \varphi \cdot (d\underline{r}/dt)\,dt \quad (\text{chain rule})$$

$$= \int_a^b \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$

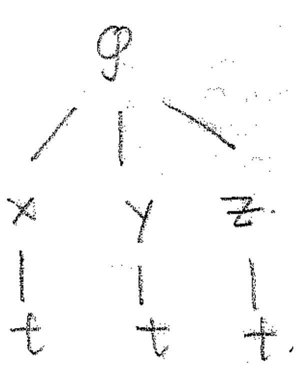
(def of $\nabla\varphi$ & $\underline{r}$ in 3-D).

$$= \int_a^b \left(\frac{\partial \varphi}{\partial x}\frac{dx}{dt} + \frac{\partial \varphi}{\partial y}\frac{dy}{dt} + \frac{\partial \varphi}{\partial z}\frac{dz}{dt}\right) dt$$

(def dot product)

$$= \int_a^b \frac{d\varphi}{dt}\,dt \quad (\text{chain rule})^*$$

*



$$= \int_{\varphi(\underline{r}(a))}^{\varphi(\underline{r}(b))} d\varphi$$

$t = a \Rightarrow \varphi = \varphi(\underline{r}(a))$.

$t = b \Rightarrow \varphi = \varphi(\underline{r}(b))$.

$$= \varphi \Big|_{\varphi(\underline{r}(a))}^{\varphi(\underline{r}(b))}$$

$$= \varphi(\underline{r}(b)) - \varphi(\underline{r}(a)) \quad \checkmark$$

$\Rightarrow$ Line integrals of gradient fields: $\int_C \underline{\nabla}\varphi \cdot d\underline{r}$ (*)

are "PATH INDEPENDENT"

NOTE THE INTEGRAL (*) AROUND A CLOSED PATH  C IS ZERO.

because $\underline{r}(b) = \underline{r}(a)$.

Calculus Stewart, 5th ed.

Conservative VFs, 2

Conservation of Energy

Let's apply the ideas of this chapter to a continuous force field $\mathbf{F}$ that moves an object along a path C given by $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, where $\mathbf{r}(a) = A$ is the initial point and $\mathbf{r}(b) = B$ is the terminal point of C. According to Newton's Second Law of Motion (see Section 14.4), the force $\mathbf{F}(\mathbf{r}(t))$ at a point on C is related to the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$ by the equation

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$$

So the work done by the force on the object is

$$\begin{aligned}
W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\, dt \\
&= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)]\, dt \qquad \text{(Theorem 14.2.3, Formula 4)} \\
&= \frac{m}{2} \int_a^b \frac{d}{dt} |\mathbf{r}'(t)|^2\, dt \\
&= \frac{m}{2} [|\mathbf{r}'(t)|^2]_a^b \qquad \text{(Fundamental Theorem of Calculus)} \\
&= \frac{m}{2} (|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2)
\end{aligned}$$

Therefore

$$W = \tfrac{1}{2}m|\mathbf{v}(b)|^2 - \tfrac{1}{2}m|\mathbf{v}(a)|^2 \tag{15}$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity.

The quantity $\frac{1}{2}m|\mathbf{v}(t)|^2$, that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore, we can rewrite Equation 15 as

$$W = K(B) - K(A) \tag{16}$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that $\mathbf{F}$ is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$. In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$. Then by Theorem 2 we have

$$\begin{aligned}
W &= \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} \\
&= -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] \\
&= P(A) - P(B) = K(B) - K(A)
\end{aligned}$$

Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.