CN14: Line Integration Problems

Ex. Evaluate $\int_C \underline{F}(\underline{r}) \cdot d\underline{r}$, where $\underline{F}(\underline{r}) = \underline{F}(x,y,z) =$

$= (yz,\ xz,\ xy+2z)$, & C is the line segment

from $(1,0,-2)$ to $(4,6,3)$.

<u>method 1</u>

$$\int_C \underline{F}(\underline{r}) \cdot d\underline{r} = \phi(\underline{r}(b)) - \phi(\underline{r}(a))$$ } Fundamental Thm of Line Integration

$$= \phi(4,6,3) - \phi(1,0,-2)$$

$$= \left(xyz + z^2 + K\right)\Big|_{(4,6,3)} - \left(xyz + z^2 + K\right)\Big|_{(1,0,-2)}.$$ } $\underline{F} = \underline{\nabla}\phi$: ϕ is a potential fnc for $\underline{F}$.

$$= 72 + 9 + K - (0 + 4 + K)$$

$$= 81 + K - 4 - K = \boxed{77}.$$

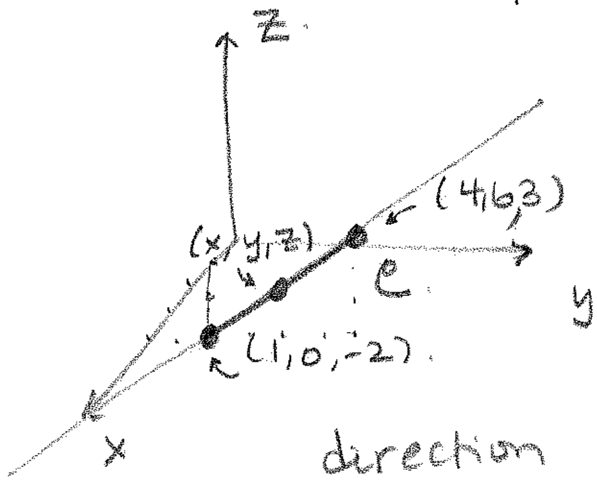
$\underline{F}$ is conservative bec. $\underline{F} = \underline{\nabla}\phi$; $\int_C \underline{F}(\underline{r}) \cdot d\underline{r}$ is path-independent.

Method 2. old method was to find

$$\int_{\mathcal{C}} \underline{F}(\underline{r})\cdot d\underline{r} \quad \text{as} \quad \int_{\mathcal{C}} (F_1, F_2, F_3)\cdot(dx, dy, dz)$$

$$= \int_{\mathcal{C}} F_1\,dx + F_2\,dy + F_3\,dz \;:\; \int_a^b \underline{F}(t)\cdot \underline{r}'(t)\,dt.$$

We need a parametric form for $\mathcal{C}$: $x = x(t)$, $y = y(t)$, $z = z(t)$.



Direction of the line is

$\underline{V} = (4-1, 6-0, 3-(-2)) = (3,6,5)$.

direction from $(1,0,-2)$ to (x,y,z) on $\mathcal{C}$ is

$(x-1, y-0, z-(-2))$, which is a multiple of $\underline{V} = (3,6,5)$

$$\Rightarrow (x-1, y, z+2) = t\,(3,6,5) = \begin{cases} x-1 = 3t \\ y = 6t \\ z+2 = 5t, \quad 0 \le t \le 1 \end{cases}$$

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$x = 1+3t, \quad y = 6t, \quad z = -2+5t.$

$\underline{r}(t) = (x(t), y(t), z(t)) = (1+3t, 6t, -2+5t).$

$\underline{r}'(t) = (3, 6, 5).$

$$\underline{F}(t) = (y(t)z(t),\ x(t)z(t),\ x(t)y(t) + 2z(t))$$

$$= (6t(-2+5t),\ (1+3t)(-2+5t),\ (1+3t)6t + 2(-2+5t))$$

$$= (30t^2 - 12t,\ 15t^2 - t - 2,\ 18t^2 + 16t - 4)$$

$$\int_0^1 \underline{F}(t) \cdot \underline{r}'(t)\,dt = \int_0^1 [90t^2 - 36t + 90t^2 - 6t - 12 + 90t^2 + 80t - 20]\,dt$$

$$= \int_0^1 (270t^2 + 38t - 32)\,dt = \left(\frac{\overset{90}{\cancel{270}}t^3}{\cancel{3}} + \frac{\overset{19}{\cancel{38}}t^2}{\cancel{2}} - 32t \right)\Big|_0^1$$

$$= 90 + 19 - 32 = \boxed{77}$$

Comparing method 1 & 2, how do you like the Fundamental Thm of Line Integration?

Ex Show that $\int_C 2x\sin y\,dx + (x^2\cos y - 3y^2)\,dy$ is independent of path, & evaluate it along any path from $(-1,0)$ to $(5,1)$.

We have $\int_C \underline{F}(\underline{r})\cdot d\underline{r}$, where $\underline{F}(\underline{r}) = \underline{F}(x,y) = (2x\sin y,\ x^2\cos y - 3y^2)$,

$d\underline{r} = (dx, dy)$.

For path independence, show $\underline{F} = \nabla\phi$: ✓

(a) $2x\sin y = \dfrac{\partial\phi}{\partial x}$; $x^2\cos y - 3y^2 = \dfrac{\partial\phi}{\partial y}$ (b).

Int (a) w.r.t. x: $\phi(x,y) = x^2\sin y + g(y)$. sub. into (b):

$\dfrac{\partial}{\partial y}\left[x^2\sin y + g(y)\right] = x^2\cos y - 3y^2$.

$\Leftrightarrow \quad x^2\cos y + g'(y) = x^2\cos y - 3y^2 \Leftrightarrow g'(y) = -3y^2$.

if $g(y) = -y^3$. So a potential fnc $\phi = x^2\sin y - y^3$

So $\int_C \underline{F}(\underline{r})\cdot d\underline{r} = \phi(5,1) - \phi(-1,0) = \boxed{25\sin(1) - 1}$.