CN 13:

Potential functions.

If a velocity field $\underline{v}$ can be written as the gradient of some scalar function u, we called the flow a potential flow & u was a potential function for $\underline{v}$.

A vector field $\underline{F}$ that can be written as $\underline{\nabla}\phi$ is also called a conservative vector field.

This use of "conservative" is new compared with our previous discussion of conservation of mass.

not all vector fields are conservative, but many in physics are.

If $\underline{F}$ is conservative, how can we write it as $\underline{\nabla}\phi$? How can we find ϕ?

It turns out $\underline{F} = (yz, xz, xy+2z)$ is conservative. It has a potential functions

$$\phi = xyz + z^2 + K.$$

Verify that $\underline{F} = \underline{\nabla}\phi$!

$$\underline{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$= (yz, xz, xy+2z) = \underline{F}(x,y,z) \checkmark$$

$\underline{\nabla}\times\underline{F} = \vec{0}$

$\exists\phi \ni \underline{\nabla}\phi = \underline{F}$

How did I find the potential func ϕ for $\underline{F}$?

$$\underline{F}(x,y,z) = (yz,\ xz,\ xy+2z)$$

Seek $\phi(x,y,z)$ so that $\underbrace{\frac{\partial \phi}{\partial x} = yz}_{ⓐ}$, $\underbrace{\frac{\partial \phi}{\partial y} = xz}_{ⓑ}$, $\underbrace{\frac{\partial \phi}{\partial z} = xy+2z}_{ⓒ}$

ⓐ $\frac{\partial \phi}{\partial x} = yz$

Integrate with respect to x (w.r.t. x)

$$\phi(x,y,z) = yzx + \underbrace{g(y,z)}_{\text{arbitrary "constant" w.r.t. } x}$$

ⓑ $\frac{\partial \phi}{\partial y} = xz$

$$\frac{\partial}{\partial y}\left[yzx + g(y,z)\right] = zx + \frac{\partial}{\partial y} g(y,z) = xz$$

$$\frac{\partial}{\partial y} g(y,z) = 0 \quad \text{int. w.r.t. } y:$$

$$g(y,z) = \underbrace{h(z)}_{\text{"constant" w.r.t. } y}$$

Now we can write $\phi(x,y,z) = yzx + h(z)$.

(c) $\frac{\partial \phi}{\partial z} = xy + 2z$

$$\frac{\partial}{\partial z}\left[yzx + h(z) \right] = yx + h'(z) = xy + 2z$$

$h'(z) = 2z$ int. w.r.t. z:

$h(z) = z^2 + K$

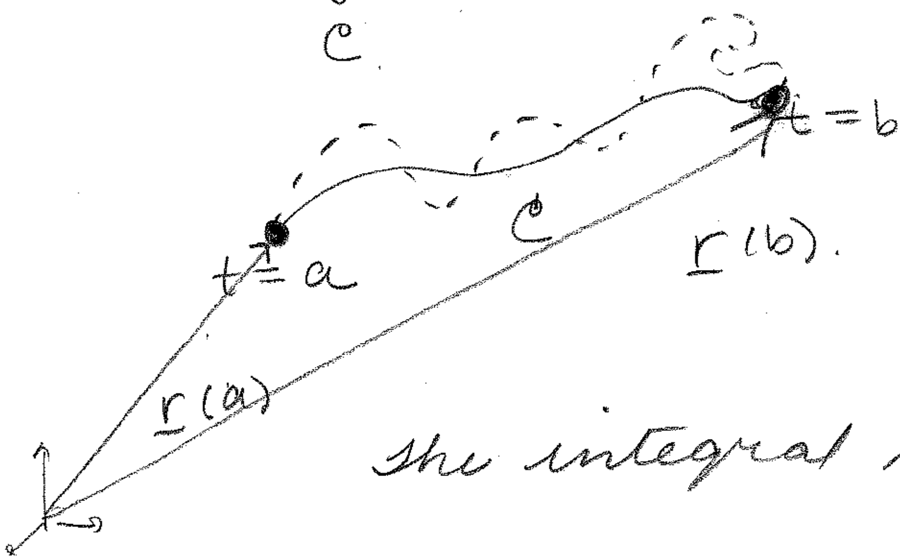
Now we have $\boxed{\phi(x,y,z) = yzx + z^2 + K}$

$\nabla \phi = (yz, xz, xy + 2z)$ ✓

Conservative vector fields allow simplification of line integration:

If $\underline{F}(x,y,z) = \underline{F}(\underline{r}) = \underline{\nabla}\phi(\underline{r})$ then

$$\int_C \underline{F}(\underline{r}) \cdot d\underline{r} = \phi(\underline{r}(b)) - \phi(\underline{r}(a))$$



Fundamental Theorem of Line Integration.

the integral is path independent.