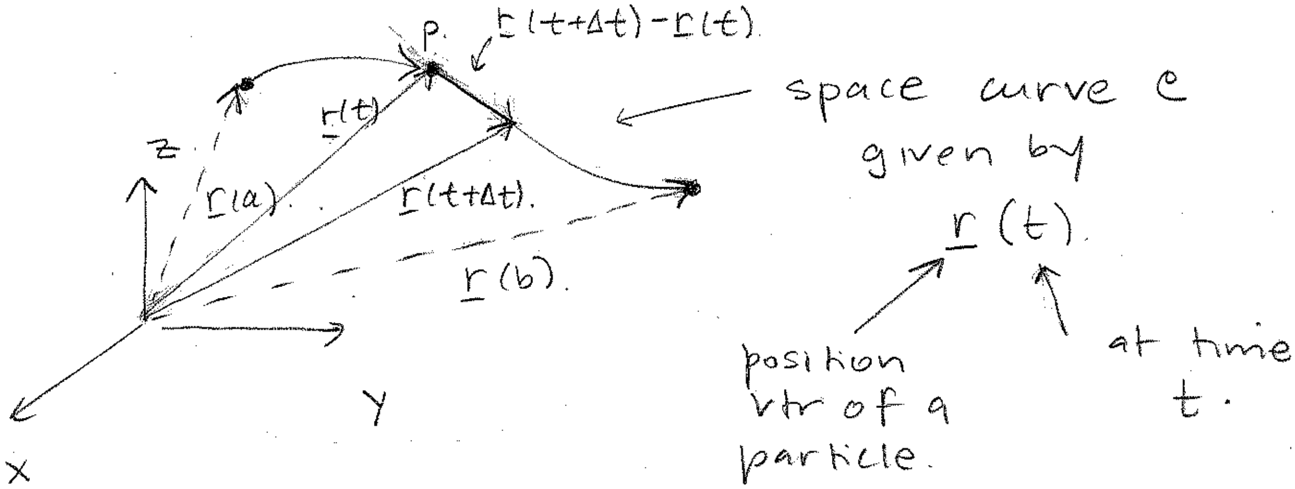
CN 12:

Line Integration

Aim: Particle is moved along a curve C in 3-D from $\underline{r}(a)$ to $\underline{r}(b)$ by the force field $\underline{f}(\underline{r})$. What is the work done on the particle by the force field?



Background info

- $\underline{r}'(t)$ is a tangent vector to C at P. It measures the rate of change of the position vector at P.
- If $\underline{r}(t)$ is a position vtr, $\underline{r}'(t)$ is a velocity vtr.
- $\|\underline{r}'(t)\|$ is particle speed.

  Recall $\| \langle a, b, c \rangle \| = \sqrt{a^2+b^2+c^2}$

Call the distance the particle travels along $\mathcal{C}$ between time t & time $t+\Delta t$ $\boxed{\Delta s}$.

$\Delta s \approx$ rate $\times \begin{pmatrix}\text{time}\\ \text{elapsed}\end{pmatrix} = \|\underline{r}'(t)\|\,\Delta t$

length of $\mathcal{C}$ from $t=a$ to $t=b$ is

$$L \approx \sum \Delta s \xrightarrow[\text{length of subint} \to 0]{\#\text{ of subintervals} \to \infty} \boxed{L = \int_a^b \|\underline{r}'(t)\|\,dt}$$

arclength.

Another notation:

$$\underline{r}(t) = (x(t),\, y(t),\, z(t)) \rightarrow$$
$$\underline{r}'(t) = (dx/dt,\, dy/dt,\, dz/dt)$$
$$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}\; dt$$

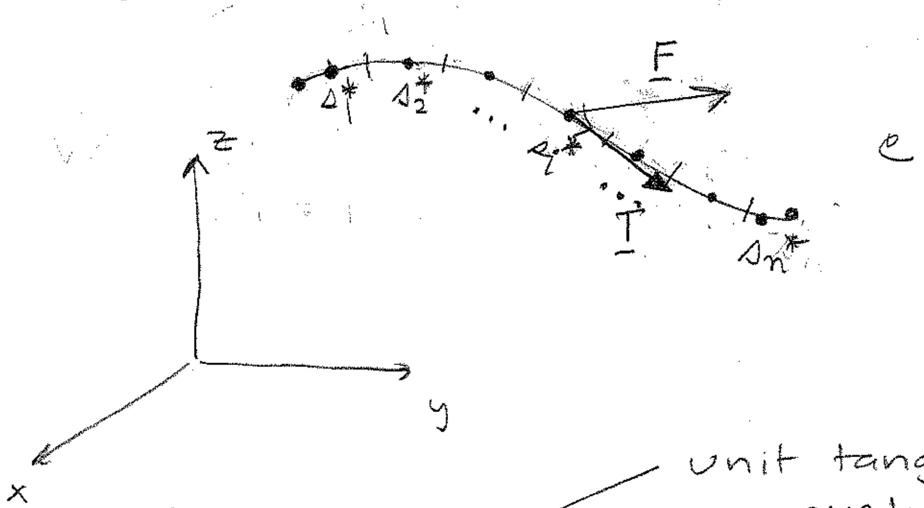
$$\boxed{\text{Arclength } \underline{\text{function}}: \quad s(t) = \int_a^t \|\underline{r}'(\tau)\|\, d\tau}$$

Distance along $\mathcal{C}$ that the particle travels when time increases from a to t

Note $\frac{ds}{dt} = \|\underline{r}'(t)\|$ by FTC. (+).

what is the work done by force field

$\underline{F}(\underline{r})$ in moving a particle along C?



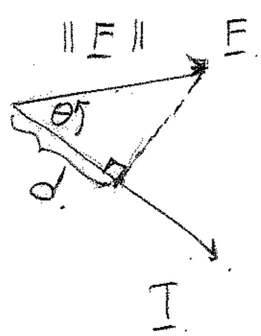
unit tangent vtr

evaluate at point in subinterval.

$$W \approx \sum_{i=1}^{n} \underbrace{(\underline{F} \cdot \underline{T})}_{d}(\Delta_i^*)\,\Delta s_i$$

d = component of force in the unit tangent direction.

why does dot product give the comp't d ?



$$\cos\theta = \frac{d}{\|\underline{F}\|} \rightarrow d = \|\underline{F}\|\cos\theta \rightarrow$$

$$d = \|\underline{F}\|\,\underbrace{\|\underline{T}\|}_{=1}\cos\theta$$

unit tangent $\underline{T}$

$$= \underline{F} \cdot \underline{T}.$$

$$W = \int_0^L (\underline{F} \cdot \underline{T})(s)\,ds.$$ in terms of t ...

$$W = \int_a^b \underline{F}(s(t)) \cdot \frac{\underline{r}'(t)}{\|\underline{r}'(t)\|} \frac{ds}{dt}\, dt$$

$\|\underline{r}'(t)\|$ by (†).

$$W = \int_a^b \underline{g}(t) \cdot \underset{d\underline{r}/dt}{\underline{r}'(t)}\, dt$$

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shorthand $\quad W = \int_C \underline{f}(\underline{r}) \cdot d\underline{r}$

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ex Determine the work done on a particle that is moved on a curve

$x(t)$ $y(t)$ $z(t)$

$C: \underline{r}(t) = (t^2, t, t^{-1})$ feet

from $(1,1,1)$ to $(4,2,\frac{1}{2})$ by a force field

$\underline{f}(x,y,z) = (yz, xz, xy)$ in pounds

$$\int_{a?}^{b?} \underline{\mathcal{F}}(t) \cdot \underline{r}'(t)\, dt.$$

$$\underline{r}'(t) = (2t, 1, -t^{-2})$$

$$\underline{\mathcal{F}}(t) = \underline{f}(x(t), y(t), z(t)) = \big(y(t)z(t),\ x(t)z(t),\ x(t)y(t)\big)$$

$$= (t\,t^{-1}, t^2 t^{-1}, t^2 t) = (1, t, t^3).$$

$$\underline{\mathcal{F}}(t) \cdot \underline{r}'(t) = (1, t, t^3) \cdot (2t, 1, -t^{-2})$$

$$= 2t + t - t = 2t.$$

$$\int_1^2 2t\, dt. = t^2 \Big|_1^2 = 4 - 1 = \boxed{3} \text{ ft-lbs.}$$

$(1,1,1) = (t^2, t, \frac{1}{t})$ at $t = 1$

$(4,2,\frac{1}{2}) = (t^2, t, \frac{1}{t})$ at $t = 2$