CN10 Stokes' & Divergence Theorems

Our derivation of Laplace's eq. for an ideal fluid relied on the ideas of

- velocity potential:

$$\underline{v} = \nabla u$$ ← potential function

↑ velocity field

and

- fluid conservation

Two questions to lead us into the two fundamental theorems of vector calculus (which motivate our study of line and surface integrals are:

Q1: What is a physical meaning of the potential function u if $\nabla u = \underline{v}$, & $\underline{v}$ is a velocity field?

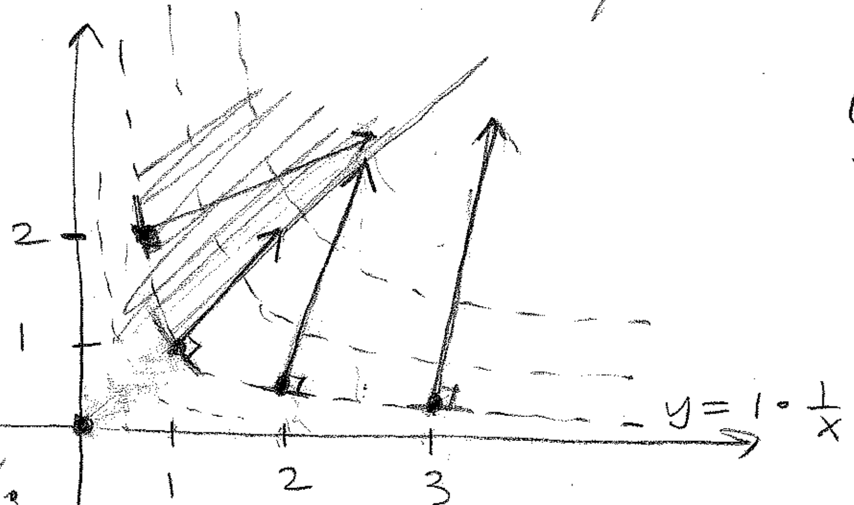
Q2: If divergence of the flow rate div $\underline{w}$ gives the amt. of fluid leaving a unit vol. per unit time, how can we interpret the curl?

ex 1 p185

If the potential is $u(x,y) = xy$, then the velocity field is $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = (y, x)$

Let's graph curves $u =$ constant (the equipotential curves in the first quadrant.

If $u(x,y)$ is a surface, then $c = u(x,y)$ are the curves of constant altitude. The gradient points in the direction of steepest ascent.



$u = c$

$xy = c$

$y = c \cdot \frac{1}{x}$

Let's superimpose the graph of the velocity field.

A1: Level curves of the potential function u give the equipotential curves.

X	Y	$\underline{v} = (y, x)$
1	1	$(1, 1)$
2	$\frac{1}{2}$	$(\frac{1}{2}, 2)$
3	$\frac{1}{3}$	$(\frac{1}{3}, 3)$
$\frac{1}{2}$	2	$(2, \frac{1}{2})$
0	0	$(0, 0)$

← fluid is motionless at origin

The fluid is carried along <u>streamlines</u> that track the velocity vectors.

This is flow past a 45° wedge.

This is an irrotational flow. It has no whirlpool, eddy, vortex.

A2: vorticity = curl $\underline{v}$ = $\dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_1}{\partial y}$

Flow is irrotational if curl $\underline{v} = 0$ at every point (x, y).

Note: The curl is really a vector, & $\dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_1}{\partial y}$ is only one of its components. For 2-D flow (xy-plane), the other components are zero, so we call $\dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_1}{\partial y}$ the curl until we go to 3-D.

Another way to express the absence of rotation is:

the velocity has zero circulation

($\int_{C} v_1\,dx + v_2\,dy = 0$

around any closed path C.)

Even without knowing how to compute the line integral we can see that checking that $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$ is zero is easier than checking for zero circulation around EVERY closed path C.

In our ex. show $\underline{v} = (y, x)$ is an irrot flow:

curl $\underline{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 1 - 1 = 0$ ✓

We need to practice line int's also.

On the ex1 what is the direction of most rapid change for u?

ans: $\underline{\nabla} u = (y, x)$.

What is the magnitude of most rapid change?

ans: $\| \underline{\nabla} u \| = \sqrt{y^2 + x^2}$. *

Note this is also the distance from (x,y) to the origin for this ex.

In this ex, fluid is flowing faster the farther it is from the origin.

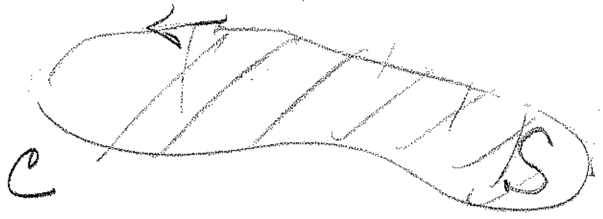
* magnitude of a vector (a,b) is denoted $\|(a,b)\|$ or $|(a,b)|$ & is calculated $\sqrt{a^2+b^2}$.

Stokes' Thm allows us to relate zero circulation & zero vorticity. It says

$$\int_C v_1 dx + v_2 dy = \iint_S \text{curl}\, v \, dx\, dy.$$

all loops C have zero circulation when curl $\underline{v}=0$ & vice versa.

We need to learn surface integration



Let's relate irrotational flows to potential flows:

Potential flows are irrotational! (not just in our ex. $u = xy$).

For a potential flow, the velocity field $\underline{v}$ can be expressed as the gradient of a potential function u.

In this case, flow is irrotational:

Check that curl $\underline{v} = 0$:

$$\text{curl } \underline{v} = \text{curl}(\underline{\nabla} u) = \text{curl}\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y \partial x}$$

$= 0$ if u has continuous first partial derivatives.

In words, the curl of a gradient is zero. If $\underline{v}$ comes from a potential u ($\underline{v} = \nabla u$), then the flow has zero vorticity.

The opposite is also true: If curl $\underline{v} = 0$, and the region in question is simply connected (does not have holes), then $\underline{v}$ is the gradient of a potential.

So in a region without holes, irrotational flow and potential flow are the same.

Ex 2 Is flow irrotational for the potential

$$u = x^2+y^2 \; ?$$

$$\underline{v} = \underline{\nabla} u = \left(\frac{\partial}{\partial x}(x^2+y^2), \frac{\partial}{\partial y}(x^2+y^2)\right)$$

$$= (2x, 2y)$$

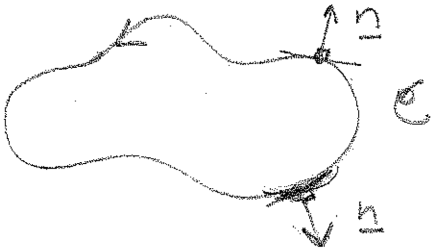
$$\text{curl } \underline{v} = \frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(2x)$$

$$= 0 - 0$$

$$= 0 \checkmark$$

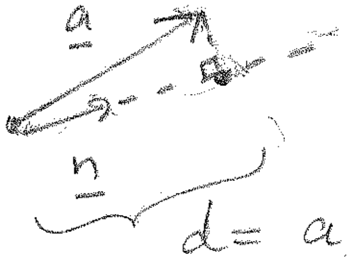
The Divergence Thm in 2-D is

$$\iint_S \text{div}\, \underline{w}\, dx\,dy = \int_C \underline{w}\cdot\underline{n}\, ds \qquad \text{p187 (11)}$$



$\underline{n}(x,y)$ is the unit outward normal — a unit vtr pointing out of the region, perpendicular to the curve C.

$\underline{w}\cdot\underline{n} = (w_1,w_2)\cdot(n_1,n_2) = w_1n_1+w_2n_2$ is the rate of flow that goes thru C. In fact, this is how dot product $\underline{a}\cdot\underline{n}$ can be interpreted: as the component of $\underline{a}$ in the direction of the unit vector $\underline{n}$, i.e., how far in the direction of $\underline{n}$ do you have to go if you're trying to reach the tip of $\underline{a}$?



$d = \underline{a}\cdot\underline{n}$ = component of $\underline{a}$ in the direction of unit vector $\underline{n}$.

In the Divergence Thm

in $\left(\iint_S \operatorname{div} \underline{w}\, dx\,dy = \int_C \underline{w} \cdot \underline{n}\, ds\right)$, the length

is measures the arc the flow passes through, so integrating $\underline{w} \cdot \underline{n}\, ds$ gives the ~~total~~ (net) outward flow — the flow out minus the flow in.

The Divergence Thm matches this outward flow across the curve C to the flow into/out of the sinks/sources in the region enclosed by the curve C

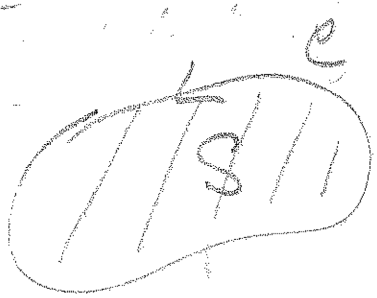
Suppose $\operatorname{div} \underline{w} = 0$: There are no sources/sinks. Then the $\iint_S \operatorname{div} \underline{w}\, dx dy = 0$. So the Divergence Thm says the $\underbrace{\text{net flow across } C}_{\int_C \underline{w} \cdot \underline{n}\, ds}$ is zero.

If a region has no sources or sinks, then the net flow through every closed curve is zero.

It turns out the opposite is also true:

If the net flow across every closed curve is zero, then the region has no sources or sinks:

$$\oint_{\textcircled{C}} \underline{w}\cdot\underline{n}\,ds = 0 \Rightarrow \underbrace{\iint_S \operatorname{div}\underline{w}\,dx\,dy = 0}$$



Because this is true for every S, the integrand div $\underline{w}$ must be zero.

The global statement of no net flow out of / into a region S and the local statement of no divergence at a point (x, y) both express conservation of mass: mass is neither created nor destroyed.

Back to Ex 1:

Given the potential $u = xy$, is mass conserved for the incompressible flow $\underline{w} = c\underline{v}$? $\iff$ Is $\operatorname{div} \underline{w} = 0$?

$$
\begin{aligned}
\operatorname{div} \underline{w} = \operatorname{div}(c\underline{v}) &= \operatorname{div}(c\underline{\nabla} u). \\
&= \operatorname{div}\left(c\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)\right) \\
&= \operatorname{div}(c(y, x)) \\
&= \operatorname{div}(cy, cx) \\
&= \underline{\nabla} \cdot (cy, cx) \\
&= \frac{\partial}{\partial x}(cy) + \frac{\partial}{\partial y}(cx) \\
&= 0 + 0 = 0 \ \checkmark
\end{aligned}
$$

mass is conserved.

If $\operatorname{div} \underline{w} = 0$, then $\iint_S \operatorname{div} \underline{w} \, dx dy = 0$,

So the Divergence Theorem says

$$0 = \underbrace{\iint_S \operatorname{div} \underline{w} \, dx \, dy}_{\text{no sources or sinks in the region}} = \underbrace{\int_C \underline{w} \cdot \underline{n} \, ds}_{\text{no flow across the boundary of the region}}$$

note for an incompressible flow $\underline{w} = c\underline{v}$,

$$\underbrace{\operatorname{div} \underline{w} = 0} \iff \underline{\underline{\operatorname{div} \underline{v} = 0}}:$$

For an incompressible flow $\underline{w} = c\underline{v}$, the flow is divergence free, is equivalent to the velocity is divergence free.

Let's reinforce the interpretation of divergence.

For incompressible flow,

$$\operatorname{div} \underline{V} = 0 \iff \operatorname{div}(\underline{\nabla} u) = 0$$

$$\iff \nabla \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = 0$$

$$\iff \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\iff$ u satisfies Laplace's eq.

We explained this on Derivation of Laplace, p9 as net flow through any closed curve is zero.

Fluid can travel with constant density along "streamlines" orthogonal to equipotential curves $u(x,y) = c$.

Back to Ex 2:

Given the potential function $u = x^2 + y^2$, do we have conservation of mass for the uncompressible flow $\underline{w} = c\underline{v}$? $\iff \operatorname{div} \underline{v} = 0$?

$$\operatorname{div} \underline{v} = \operatorname{div}(\underline{\nabla} u) = \operatorname{div}\left(\frac{\partial}{\partial x}(x^2+y^2), \frac{\partial}{\partial y}(x^2+y^2)\right)$$

$$= \operatorname{div}(2x, 2y) = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y)$$

$$= 2 + 2$$

$$= 4.$$

mass is not conserved.

We have to create fluid to maintain a steady flow. The fluid has to be created everywhere (at a uniform rate of quadrupling per unit time). note $\operatorname{div} \underline{w} = 4c$.