CN9 A derivation of Laplace's eq. to lead us into a discussion of line and surface integrals (vector calculus or calculus of vector fields).

Laplace's eq. $\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}} = 0$

and potential flow (today) § 3.3 p. 182.
Later we'll use Fourier series and complex variables in solving the eq. analytically.
Ch. 5 discusses numerical solutions.
Today we look at a particular context for Laplace's eq:

Ideal fluids - no viscosity to dissipate energy.

Steady, 2-D flow.

a variable $\underline{V} = (V_1(x,y), V_2(x,y))$ gives the direction and speed of the flow at each point (x,y).

It is a vector field: a function that assigns vectors $(V_1(X,y), V_2(X,y))$ to points (x,y) in the plane.

It is a 2-D velocity field.

Because the flow is steady (time-independent),

- The velocity should come from a potential $u(x,y)$: It should be expressable as $\underline{V} = \nabla u = \text{grad } u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$.

$\underline{V}$ is a "potential flow"

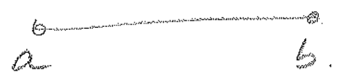
- The fluid should be conserved — neither created nor destroyed, provided there are no fluid sources or sinks.

To express these properties as differential eqns. (DEs) is a perfect application of two fundamental theorems of vector calculus (calculus on vector fields), involving

- line integrals, which can be used to find the work done by a force field in moving an object along a curve, and
- surface integrals, which can be used to find the rate of fluid flow across a surface.

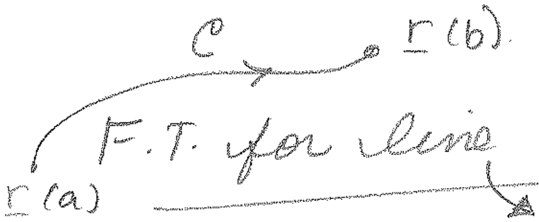
These two kinds of integrals relate to single, double, and triple integrals from our calc sequence via higher dimensional versions of the Fundamental Theorem of Calculus (FTC).

what will these thms look like? deriv's Kapl. 4



F.T.C.

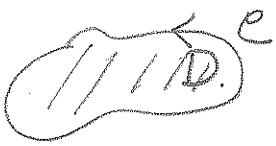
$$\int_a^b F'(x)\,dx = F(b) - F(a)$$



F.T. for line int's

$$\int_C \underline{\nabla} f \cdot d\underline{r} = f(\underline{r}(b)) - f(\underline{r}(a)).$$

Green's thm

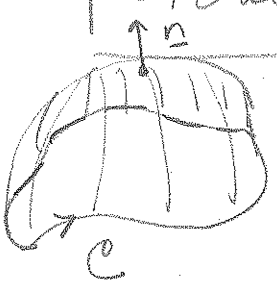


dble int

line int

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy.$$

$\boxed{\text{Stokes' Thm}}$



surface int

line int.

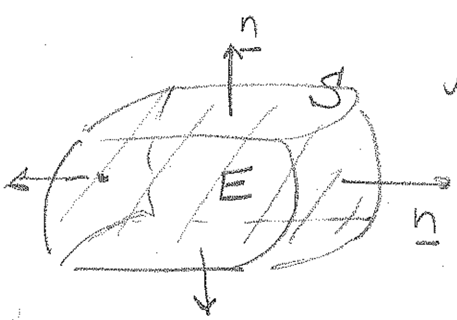
$$\iint_S \text{curl}\, \underline{F} \cdot d\underline{S} = \int_C \underline{F} \cdot d\underline{r}$$

$\boxed{\text{Divergence Thm}}$

triple int

surf int.

$$\iiint_E \text{div}\, \underline{F}\, dV = \iint_S \underline{F} \cdot d\underline{S}$$

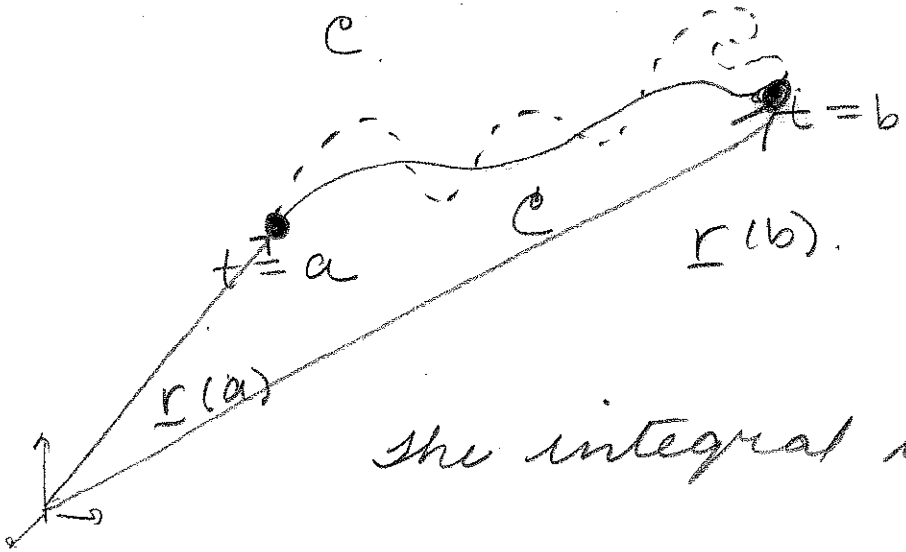


We need to know dot product ($\cdot$), grad, curl & divergence (div) operations.

Conservative vector fields allow simplification of line integration:

If $\underline{F}(x,y,z) = \underline{F}(\underline{r}) = \underline{\nabla}\phi(\underline{r})$ then

$$\int_{C} \underline{F}(\underline{r}) \cdot d\underline{r} = \phi(\underline{r}(b)) - \phi(\underline{r}(a))$$



Fundamental Theorem of Line Integration.

The integral is path independent.

The laws of vector calculus come from physical principles.

Q: Why do potential flows for which we are deriving a PDE model arise in many applications?

(Remember a velocity field $\underline{v}$ describe a potential flow if $\underline{v} = \underline{\nabla} u$.)

A: Because $\underline{\nabla} u$ points in the direction of ~~fastest~~ change (steepest increase) of u.

($-\underline{\nabla} u$ points in the direction of most rapid decrease of u.)

Ex (a) In the atmosphere, air (a fluid) flows in the direction of steepest pressure drop. $\underline{v}(x,y) \propto -\underline{\nabla} p$ ← pressure

(b) Heat flows in the direction of steepest temperature drop. heat flux → $q \propto -\nabla u$ — temp dennis kaye, 6

what are the dots and triangles?

Definitions of some operators:

<u>dot product</u> $(u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2$

<u>gradient</u> $\text{grad} = \underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

$$\underline{\nabla}\, u(x,y) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

<u>divergence</u> $\text{div}\, \underline{w} = \underline{\nabla} \cdot \underline{w} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (w_1, w_2)$

$$\text{div}\, \underline{w} = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}$$

<u>curl in 2-D</u> $\text{curl}\, \underline{v} = \text{curl}(v_1, v_2)$

$$\text{curl}\, \underline{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$$

We'll practice these operations & give physical interpretations.

To continue our derivation of Laplace's eq., interpret the velocity field $\underline{v}$ times mass density c (often written ρ) as the momentum density or flow rate $\underline{w}$.

$$\underline{w} = c\,\underline{v}.$$

What are the units on flow rate $\underline{w}$?

$$[\underline{w}] = [c\,\underline{v}] = [c][\underline{v}] = \frac{m}{l^3}\cdot\frac{l}{t} = \frac{m}{l^2 t}$$

mass per unit area (l^2) per unit time t.

mass flow across a unit area per unit time.

So $\operatorname{div}\underline{w} = \operatorname{div} c\underline{v} = \frac{\partial}{\partial x}(c v_1) + \frac{\partial}{\partial y}(c v_2)$.

$$[\operatorname{div}\underline{w}] = \frac{\frac{m}{l^2 t}}{l} = \frac{m}{l^2 t}\cdot\frac{1}{l} = \frac{m}{l^3 t}$$

mass flow from a unit volume per unit time.

If there are no sources or sinks in the volume, mass is conserved: $\operatorname{div}\underline{w} = 0$.

Consider f to be a certain amount of mass leaving per unit volume per unit time.
If there's a source, we write:

$$\underbrace{\operatorname{div} \underline{w}}_{\text{amount leaving}} = \underbrace{-f}_{\text{is negative means mass is entering}} \quad (f > 0)$$

If there's a sink,

$$\underbrace{\operatorname{div} \underline{w}}_{\text{amount leaving}} = \underbrace{f}_{\text{is positive}} \quad (f > 0)$$

$$\begin{aligned}
\operatorname{div}(\underline{w}) &= \operatorname{div}(c\underline{V}) = \underline{\nabla} \cdot (c\underline{V}) = \underline{\nabla} \cdot (c\,\underline{\nabla} u) \\
&= \underline{\nabla} \cdot \left(c \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \right) = \underline{\nabla} \cdot \left(c\frac{\partial u}{\partial x}, c\frac{\partial u}{\partial y} \right) \\
&= \frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y}\left(c\frac{\partial u}{\partial y} \right) \quad \text{constant } c \Rightarrow \\
&= c\frac{\partial^2 u}{\partial x} + c\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

$$\operatorname{div}(\underline{w}) = -f \iff \frac{c}{c}\boxed{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = -\frac{f}{c}} \quad \text{POISSON'S EQ}$$

For incompressible fluids without sources or sinks ($f=0$), the eq is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{Laplace's eq.}$$

$$\left(\operatorname{div} \underline{w} = 0 \iff \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0 \quad \text{"continuity eq"} \right)$$

Laplace's eq in other notations:

$$\operatorname{div} \underline{\operatorname{grad}}\, u = \underline{\nabla} \cdot (\underline{\nabla} u) = \underline{\nabla}^2 u = \Delta u$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

net flow through any closed surface is zero.