CN7: A taste of Sturm-Liouville theory

other applications require an extra term in the model:

$$-\frac{d}{dx}\left(\underline{c}\,\frac{du}{dx}\right)+\underline{q}\,u = f$$

← or sometimes λu.

c, q, f are specified

Seek $\underline{u(x)}$, subject to boundary conditions ($BC\underline{s}$)

When combined w/ certain kinds of $BC\underline{s}$, the DE & BCs form a Sturm-Liouville problem.

This is a 1-D, **second**-order, ~~constant~~-coefficient ODE

In quantum theory it is Schroedinger's eq.

When posed on a sphere, it is Legendre's eq.

It can describe oscillations of a drum (Bessel's eq.)

Solutions in some of these settings are "special functions." Here we have exponential solutions.

$-\frac{d}{dx}\left(c\,\frac{du}{dx}\right)+qu=0$ has exponential solutions,

Verify by substitution that $u=e^{x\sqrt{\frac{q}{c}}}$, $u=e^{-\sqrt{\frac{q}{c}}\,x}$

satisfies the ODE:

$$\frac{du}{dx}=e^{-\sqrt{\frac{q}{c}}\,x}\cdot -\sqrt{\frac{q}{c}} \longrightarrow c\,\frac{du}{dx}=-c\cdot\sqrt{\frac{q}{c}}\;e^{-\sqrt{\frac{q}{c}}\,x}$$

$$\frac{d}{dx}\left(-c\cdot\sqrt{\frac{q}{c}}\;e^{-\sqrt{\frac{q}{c}}\,x}\right)=-c\,\underline{\sqrt{\frac{q}{c}}}\;e^{-\sqrt{\frac{q}{c}}\,x}\cdot\underline{-\sqrt{\frac{q}{c}}}$$

$$-\frac{d}{dx}\left(c\,\frac{du}{dx}\right)=-\cancel{c}\cdot\frac{q}{\cancel{c}}\;e^{-\sqrt{\frac{q}{c}}\,x}$$

$$-\frac{d}{dx}\left(c\,\frac{du}{dx}\right)+qu=-q\,e^{-\sqrt{\frac{q}{c}}\,x}+q\,e^{-\sqrt{\frac{q}{c}}\,x}=0 \checkmark$$

How do we find these exponential solutions?

Seek a solution of the form $u = e^{ax}$ to satisfy $-cu'' + qu = 0$.

Determine the value(s) of the constant a.

$$u = e^{ax} \rightarrow u' = e^{ax} a \rightarrow u'' = e^{ax} a^2$$

Sub into the DE: $-ca^2 e^{ax} + q e^{ax} = 0$

$$\frac{\cancel{e^{ax}}(-ca^2 + q)}{\cancel{e^{ax}}} = \frac{0}{e^{ax}}$$

$$-ca^2 + q = 0$$

$$+ca^2 = +q$$

$$\rightarrow a^2 = \frac{q}{c}$$

$$\rightarrow a = \pm\sqrt{\frac{q}{c}}$$

$u = e^{ax}$

$u = e^{\sqrt{\frac{q}{c}}x}$ ✓

$u = e^{-\sqrt{\frac{q}{c}}x}$ ✓

The complete solution is a combination

$$u = d_1 e^{\sqrt{\frac{g}{c}}\, x} + d_2 e^{-\sqrt{\frac{g}{c}}\, x}, \quad \text{e.g.,} \ 0 \le x \le 1 .$$

Determine the two constants d_1 & d_2 by apply two boundary conditions (BC's) (or two initial conditions IC's).

Ex. 1 $u(0) = A$, $u(1) = B$.

Apply condition (1): $u(0) = A$: $u(0) = d_1 e^0 + d_2 e^0 = A$

2 eqns, 2 unknowns d_1, d_2.

$$d_1 + d_2 = A. \qquad (1)$$

Apply condition (2): $u(1) = B$; $u(1) = d_1 e^{\sqrt{\frac{g}{c}}} + d_2 e^{-\sqrt{\frac{g}{c}}} = B$

$$d_1 e^{\sqrt{g/c}} + d_2 e^{-\sqrt{\frac{g}{c}}} = B \qquad (2)$$

(1) $\rightarrow d_2 = A - d_1$. sub into (2): $d_1 e^{\sqrt{\frac{g}{c}}} + (A - d_1) e^{-\sqrt{\frac{g}{c}}} = B$

(3) $\longrightarrow$ Collect d_1 terms, and solve for d_1.

$$d_1\left(e^{\sqrt{g/c}} - e^{-\sqrt{g/c}}\right) = B - Ae^{-\sqrt{g/c}}$$ A+α

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$$d_1 = \frac{B - Ae^{-\sqrt{\frac{g}{c}}}}{e^{\sqrt{\frac{g}{c}}} - e^{-\sqrt{\frac{g}{c}}}}$$

(3) $d_2 = A - d_1$: $\quad d_2 = A - \dfrac{B - Ae^{-\sqrt{\frac{g}{c}}}}{e^{\sqrt{\frac{g}{c}}} - e^{-\sqrt{\frac{g}{c}}}}$

So solution to the BVP $\quad -cu'' + gu = 0,\ 0 \le x \le 1$

$u(0) = A,\ u(1) = B$

is

$$\boxed{\begin{gathered} u(x) = d_1 e^{\sqrt{\frac{g}{c}}x} + d_2 e^{-\sqrt{\frac{g}{c}}x}, \text{ where} \\ d_1 = \frac{B - Ae^{-\sqrt{\frac{g}{c}}}}{e^{\sqrt{\frac{g}{c}}} - e^{-\sqrt{\frac{g}{c}}}} \\ d_2 = A - \frac{B - Ae^{-\sqrt{\frac{g}{c}}}}{e^{\sqrt{\frac{g}{c}}} - e^{-\sqrt{\frac{g}{c}}}} \end{gathered}}$$