E.F. Downey / BSC Physics : PH402 Quantum I

Scherrer Chapt 3 Building The Schrö equat

Note Title 2/4/2006

1) Electric & Magnetic Field

2) $\vec{E}$ & $\vec{B}$ wave function

3) $\vec{E}$ & $\vec{B}$ wave equation

① $\vec{E}$, $\vec{B}$

② $\vec{F} = E_0 e^{i(kz-\omega t)} \hat{x} + \frac{E_0}{c} e^{i(kz-\omega t)} \hat{y}$

$\vec{F} = \vec{E} + i\vec{B}$ ← complex # keeping track of 2 linearly indep vectors

③ $\underbrace{\frac{d^2\vec{F}}{dx^2} = \mu_0\epsilon_0 \frac{\partial^2\vec{F}}{dt^2}}_{\text{wave equation}}$; $\underbrace{\vec{E}_i = E_i e^{i(kx-\omega t)} \quad B_i = \frac{E_i}{c} e^{i(kx-\omega t)}}_{\text{wave functions}}$

or prob Amplitudes

see $\frac{\partial^2 F}{\partial x^2} = i^2 k^2 =$

$\mu_0\epsilon_0 \frac{\partial^2 F}{\partial t^2} = \mu_0\epsilon_0 i^2\omega^2$

$k^2 = \mu_0\epsilon_0\omega^2$

$\sqrt{\frac{1}{\mu_0\epsilon_0}} = \frac{\omega}{k} = \frac{2\pi/T}{2\pi/\lambda} = \frac{\lambda}{T} = v$

then

prob $= |E_1 + E_2|^2$

prob(x) $= E_1^2 + E_2^2 + 2E_1E_2\cos(k(x_2-x_1))$

same idea w/ schrö to build ⇒

Wow! The E&M wave equation is also an Eigen equat of sorts

$\hat{D}_2 = \frac{d^2}{dx^2}$ &

$\hat{T}_2 = \frac{d^2}{dt^2}$

then

$\hat{D}_2\vec{F} = \frac{1}{v^2}\hat{T}_2\vec{F}$

relating spatial, D_2, and time features

$[\text{space}]\vec{F} = c[\text{time}]\vec{F}$

Find me an $\vec{F}$ that maintains these features

wave

e^-s

same thing.

e^-

1) e^- Field

2) e^- wave function

3) e^- wave equation!

+ deBye got $\lambda_{e^-} = \frac{h}{p_e}$ to solve Bohr atom stable-state problem

Led Schrö to talk about wave-like Behavior

& DeBye to insist --- where is the wave equat?

So we start by saying, assuming, and wave funct = prob Ampt.

$\Psi_{tot} = \Psi_a + i\Psi_B$

Like $\vec{F}$ except not a vector

⇒'s Linearly indep superposition state functs ≠ dis vector direction

$= |\text{dead}\rangle + i|\text{alive}\rangle$

w/ this, we will argue for a equation for the e^-

Try $\Psi_a = \Psi_0 e^{i(kx-\omega t)}$

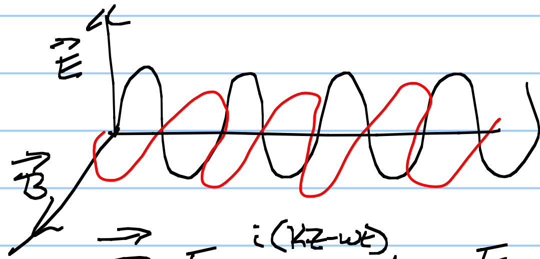
where for waves we now know $k = 2\pi/\lambda$, $\omega = 2\pi/T$, $\lambda = h/p$

& $E = h\nu = hc/\lambda$ & $p = E/c = \frac{h}{\lambda}$

NICE

Heres where we are:

1) $\vec{E}$ & $\vec{B}$ fields:



2) $\vec{E}$ & $\vec{B}$ wave function $\vec{F} = E_o e^{i(kz-\omega t)} \hat{i} + \frac{E_o}{c} e^{i(kz-\omega t)} \hat{j}$

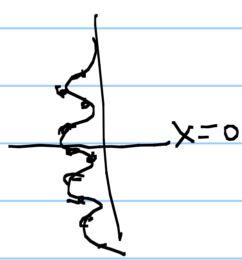
* note 2 Linearly indep solns

3) $\vec{E}$ & $\vec{B}$ wave Equation: Max's

$$\frac{d^2\vec{E}}{dx^2} = \frac{1}{v^2}\frac{d^2\vec{E}}{dt^2}$$

4) Yields probabilities in Limit of single photon

$\propto |\vec{F}(x)|^2$ such that



$$\int_{-\infty}^{+\infty} |F(x)|^2 dx = 1 \quad \therefore |\vec{F}(x)|^2 = \text{prob density}$$

here = $\frac{\text{prob}}{\text{length}}$

Now e^-

ALL particles even elephants

yield Some results as single photons, Thus

1) e^- field?

2) e^- wavefunction $(\Psi \rightarrow \Psi_1 + i\Psi_2)$

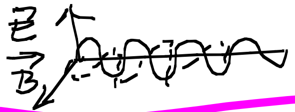
* BIG! must have 2 Linearly indep solns! But here, not vectors (ie. $\vec{E}$ & $\vec{B}$)

= Superposition State!

= |dead> + |alive> = heart of QM

3) e^- wave equation = Schrö

4) e^- prob density, $|\Psi(x)|^2 = \Psi^* \Psi = \frac{\text{prob}}{\text{length}}$

1.) E & B Field =  Electromagnetic Field

2.) E & B wavefunct → So E & M: $\vec{F} = E_0 e^{i(kz-\omega t)}\hat{x} + \frac{E_0}{c} e^{i(kz-\omega t)}\hat{y}$

$k = 2\pi/\lambda$

& $\omega = 2\pi f = \frac{2\pi}{T}$

* Note 2 - lin indep parts

since

3.) E & B wave equation →

$$\boxed{\frac{\partial^2 \vec{F}}{\partial x^2} = \mu_0\varepsilon_0 \frac{\partial^2 \vec{F}}{\partial t^2} \quad \text{or} \quad v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}}$$

contains everything about (γ) = light

massless

field, w/ $\vec{p} = \frac{h}{\lambda}$, $E = \frac{hc}{\lambda} = h\nu$

$\vec{p} = \hbar k$ $\quad E = \hbar\omega$

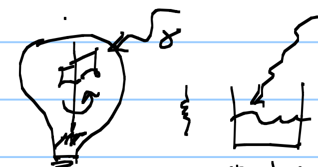
$= \hbar k c$

nice

So, lets get physical w/ the E & M wave equation.

We know $\vec{E}$ & $\vec{B}$ field carries Energy & Momentum

& it is a traveling wave @ c

* Even though a massless field! ie  γ

Bolts

where from SR.

$$E_T^2 = p^2c^2 + \underbrace{m^2c^4}_{\to 0}$$

$p = \frac{E_T}{c}$ $\qquad E_{Tot} = h\nu = \frac{hc}{\lambda} = \hbar k c = \hbar\omega$

$p = \frac{h}{\lambda} = \hbar k$

So Maxwell's equations yield

$$\cancel{i^2} k^2 \cancel{\vec{F}} = \mu_0\varepsilon_0 (\cancel{-i})^2 \omega^2 \cancel{\vec{F}} \quad \text{or that} \quad \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$$\frac{\frac{2\pi}{T}}{2\pi/\lambda} = \frac{\lambda}{T} = \text{veloc of traveling wave} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8 \text{ m/s}$$

Thus Maxwell's equations, the $\vec{E}$ & $\vec{B}$ wave equation is a
sort of eigen value problem
that looks for
a soln, the $\vec{E}$ & $\vec{B}$ wavefunction

$$\hat{D}_2 \vec{F} = \frac{1}{v^2} \hat{T}_2 \vec{F} \qquad \hat{D}_2 \equiv \frac{d}{dx^2}$$

$$\hat{T}_2 \equiv \frac{d}{dt^2}$$

That contains
Energy $\propto \omega$
& momentum $\propto K$

& is a massless wave that travels @ c.

Maxwell's Equations & its soln therefore contain all the info there is about Light!

Light = particle + wave

We will thus seek a similar equat & funct for
e^- = particle + wave

IDEA Here is

Look for sort of Hybrid Eigen problem that incorporates Both wave-like & particle-like properties.

Because Energy (Classical for now) But in either case is conserved look for

Our hybrid Eigen problem states "Find Ψ such that when you act on it w/ classical and wave-like $\hat{O}$'s that you get consistent results"

So it seems

$$\left[\begin{array}{c}\text{Classical} \\ \text{mass like} \\ \text{Like} \\ \text{properties}\end{array}\right] \Psi = \left[\text{Wave-Like}\right] \Psi$$

YES

So the goal is to find the Ψ wavefunct = state function that can contain all the particle-like and wave-like prop's

Argue: The connection of this hybrid eigen problem should be:

$$\left[\begin{array}{c}E_{TOT} \\ \text{classical} \\ \text{energy}\end{array}\right] \Psi = \left[\begin{array}{c}E_{TOT} \\ \text{is} \\ \text{wave-} \\ \text{Like}\end{array}\right] \Psi$$

is particle Like

$$\underbrace{\hat{H}_{\text{classical}}}_{\text{recognize as the Hamiltonian } \hat{O} \text{ Classically already}} \Psi = \hat{H}_{\text{wavelike}} \Psi$$

For starters: work w/ Light

after all, it is now wave + particle

Now Light is massless field, but $E_{TOT\,\gamma} = \hbar\omega = \hbar kc = \frac{hc}{\lambda}$

wave Like

particle Like

Because in 1924 de Broglie wavelength : $\lambda = \frac{h}{p}$

$$\hat{H}_{particle}\Psi = \hat{H}_{wave}\Psi$$
$$= \hbar\omega\,\Psi$$

can't really say classical here as $\gamma \neq$ classical

is $\Psi \propto e^{i(kx-\omega t)}$

then clearly

$$i\hbar\frac{\partial\Psi}{\partial t} = -i^2\omega\hbar\,\psi = h\psi$$

$$\therefore\ i\hbar\frac{\partial}{\partial t} \equiv \hat{H}_{wave\ Like}$$

$$\hat{H}_{particle\ \gamma} = E_\gamma = \frac{hc}{\lambda}$$
$$= \hbar kc$$

$\hat{H}_{particle\ \gamma}$ try

$$-i\hbar c\frac{\partial}{\partial x}$$

then

$$\hat{H}_{particle\ \gamma} = -i\hbar c\frac{\partial\Psi}{\partial x}$$
$$= -i^2\hbar ck\psi$$
$$= \hbar kc\,\Psi$$

OK

$$-i\hbar c\frac{\partial}{\partial x}(\Psi) = i\hbar\frac{\partial}{\partial t}(\Psi)$$

$$\Downarrow$$

$$-\cancel{i\hbar}c\frac{\partial}{\partial x}(\Psi) = \cancel{i\hbar}\frac{\partial\Psi}{\partial t}$$

$$c\left(-\frac{\partial\Psi}{\partial x} = \frac{\partial\Psi}{\partial t}\right)$$

$$c\left(-\cancel{i}k = -\cancel{i}\omega\right)$$

$$c = \frac{\omega}{k} = \frac{2\pi/T}{2\pi/\lambda} = \frac{\lambda}{T}$$

it has to be a photon!

Because $\frac{\lambda}{T}$ = velocity = c

So Schrö is now $\hat{H}_{\substack{\text{particle}\\ \text{classical}}} \Psi = i\hbar \frac{\partial \Psi}{\partial t}$ is really the same form.

Because $\hat{H}_{wave}$ never changes!

But

Now motivated to look for a wave-eq & wave funct that carries everything about a classical particle with mass particle, the e^-.

(free e^-) BIG

Free particle for now! so $E_{TOT} = E_K$ only & non relativistic

well $E_K = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$

So start!

$\hat{H}_{\substack{\text{classical}\\ \text{particle}}} \Psi = i\hbar \frac{\partial \Psi}{\partial t}$

ie Ham

start w/ Free (ie no E_{pot}) mass particle

NICE!

So, in some sense we are building this Hybrid eigen problem that is

$\left[\begin{smallmatrix}\text{classical}\\ \text{like}\end{smallmatrix}\right]\Psi = \left[\begin{smallmatrix}\text{wave}\\ \text{like}\end{smallmatrix}\right]\Psi$

$\hat{H} = E_K + \cancel{E_p} = E_K = \frac{1}{2} m v^2$

"massive"

or

$\hat{H} = \frac{p^2}{2m}$ why? well $p \rightarrow$ deBroglie connection

$\lambda = \frac{h}{p}$

$p = \frac{h}{\lambda}$

so $p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k$

now we are talking

since $\Psi \propto e^{i(kx - \omega t)}$

So, $p = \hbar k$

what if we define

$$\hat{p} \equiv -i\hbar \frac{\partial}{\partial x}$$

Then

$$\hat{p}\,\Psi = -i\hbar \frac{\partial}{\partial x} e^{i(kx-\omega t)} = -i^2 \hbar k\,\Psi$$

or

$$\hat{p}\,\Psi = \hbar k\,\Psi = \text{Eigen function}$$

w/ eigenvalues

= momentum

Thus

$$\hat{p} \equiv -i\hbar\frac{\partial}{\partial x} = \text{Momentum } \hat{O}.$$

Back to main issue

$$\hat{H}_{\substack{\text{free}\\\text{particle}\\\text{classical}}} = E_k = \frac{p^2}{2m} \therefore \frac{(\hat{p})^2}{2m} = \frac{(-i)^2\hbar^2\partial^2}{2m\,\partial x^2}$$

$$= \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

Free particle Solution

$$\underbrace{\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}}_{\hat{H}_{\text{classical particle-like}}}\left(\Psi_{(x,t)}\right) = \underbrace{i\hbar\frac{\partial}{\partial t}}_{\hat{H}_{\text{wave like}}}\left(\Psi_{(x,t)}\right)$$

$$\hat{H}_{\text{classical particle-like}}(\Psi) = \hat{H}_{\text{wave like}}(\Psi)$$

So we have our Schrödinger's equation for Free particle

$$\underset{\substack{\text{classical} \\ \text{Like}}}{\hat{E}_K} \Psi = \underset{\substack{\text{wave} \\ \text{Like}}}{E_{TOT}} \Psi(x,t)$$

$$\hat{H}_{classical} \Psi = i\hbar \frac{d}{dt} \Psi(x,t)$$ = Schrödingers time dep equation

Now in general:

$$\hat{H}_{classical} = \hat{E}_{TOT} = \hat{H}_{Hamiltonian} = E_K + E_p + \text{other.}$$