E.F. Deveney / BSC Physics: PH402: Q.M. Scherrer C2 ⇒ Operators H-spaces Eingen values & functions

Note Title 1/26/2006

Homework
Scherrer 2.7
2.8
2.10
2.11

Operators are HUGE in Q.M.,

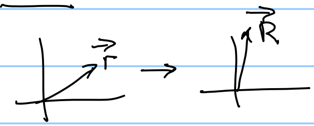
Q.M.

Heisenburg
Born, Jordon, Dirac
Matrix Q.M. Lives
in Vector Space (VS)
really complex-abstract VS

⇓

$\hat{O}$'s here rotate vectors

$\hat{O}\vec{r} = \vec{R}$



Schrödinger's wave
Q.M. Lives in
Function Space

⇓

$\hat{O}$'s change functions

ex: $D \equiv \frac{d}{dx}$, $f(x) = x^2$

$\frac{d}{dx} f(x) = 2x$ ↑ diff funct

Griffs

⇒ what ties them are that
Soln's to Schrö equations that
are Square Integrable

ie $\int_{-\infty}^{+\infty} |\Psi(x)|^2 dx = \text{Finite}$

Physical, in the Schrö-Born formalism, $|\Psi|^2 dx$ = probability

Form a Complete, Hilbert basis in F.S.

Thus
$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = \text{Finite}$

is Physical! because the prob of something must = 1

Thus non Square integrable soln's to Schrö's ≠ physical.

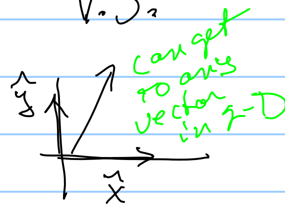
Hiesenburgs Matrix

V.S.

Schrö Wave.

F. S.

Completes:

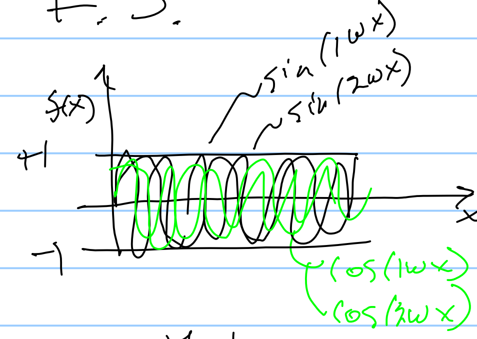


$\vec{F} = F_x\hat{x} + F_y\hat{y}$

$\hat{x}$ & $\hat{y}$ = Linearly independ Basis That "span" 2-D space. All 2-D vectors can be built as Linear combo of $\hat{x}$ & $\hat{y}$

Matrix $\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$ = row echelon reduced = 2-D

complete ⇒ complete Spans



see that

$f(x) = \sum_{n=0}^{\infty} a_n \cos n\omega_x x + b_n \sin n\omega_x x$

Fourier series

because $\cos n\omega_x x$ & $\sin n\omega_x x$ for $n = 0 \to \infty$ = Basis that "fills" all of function space & ∴ can represent any funct as linear combo of this basis

* ∞ly dimensional Basis set!

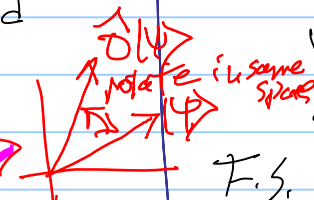
complete ⇒ is completely fills, no holes

Hilbert ⇒ in V.S. you have

$\hat{i}\cdot\hat{i} = 1$
$\hat{i}\cdot\hat{j} = 0$ } ie dot prod

⇒ Hilbert spaces are function spaces that have all the properties of V.S.

F.S. dot prod ≡ $\int a_n a_m dx = \delta_{nm}$



Dirac

CAVS $\Psi \to |\Psi\rangle$ "kets" Vectors

& $\langle\Psi|$ bra vectors

$\langle\Psi|\Psi\rangle$ = dot product

$\hat{O}$'s rotate vectors $\hat{H}|\psi\rangle$

$\hat{O}$'s change functions $\hat{H}\Psi$

Schrö → $\hat{O}\psi(x)$ = new $f(x)$

Ψ = wavefunction

= F.S. basis But in same Vector space

We'll be doing Schrö-wave-1st Q.M. so lets look @ function Ô's.

A function $f(x)=x^3$ changes #'s into new #'s

$f(2)=8$
old ↑ new

An Operator, $D \equiv \frac{d}{dx}$ changes function into new functions

ex $f(x)=x^3 \leftarrow$ old

$\frac{Df(x)}{dx} = 3x^2 \leftarrow$ new

Since Ô's will rotate vectors & change functions we will only be interested in Ô that keep us in the same "space"

so

our rotations must keep us in same V.S.

our changes to functs must keep us in sam F.S.

These Ô's are called Linear Ô's

Linear Ô's are those that

1.) $\hat{L}[f(x)+g(x)] = \hat{L}(f(x)) + \hat{L}(g(x))$

2.) $\hat{L}[c\, f(x)] = c\, \hat{L}\, f(x)$

↑ = constant.

ex: $\hat{A} \equiv [\;]^2$ and $\hat{D} \equiv \frac{d}{dx}$

try 1.) $\hat{A}[f(x)+g(x)] = [f(x)+g(x)]^2$

$= f(x)^2 + g(x)^2 + 2f(x)g(x)$

$\stackrel{?}{=} \hat{A}[f(x)] + \hat{A}[g(x)]$

$= f(x)^2 + g(x)^2$ No!

$\hat{A}$ is not a linear $\hat{O}$

what about $\hat{D}$

$\hat{D}[f(x)+g(x)] = \frac{df(x)}{dx} + \frac{dg(x)}{dx} \stackrel{?}{=} D[f(x)] + D[g(x)]$

$= \frac{df(x)}{dx} + \frac{dg(x)}{dx}$

yup!

& $\hat{D}[cf(x)] = c\frac{df(x)}{dx} \stackrel{?}{=} cD[f(x)] = c\frac{df(x)}{dx}$

yup!

$\hat{D}$ = Linear $\hat{O}$

Now, we are going to be interested in very particular functions or vectors

$\hat{O}$'s for those functions & vectors.
Linear because they keep us in the same "space"

Eigen functions (substitute eigen vectors if in V.S.)
&
Eigenvalues

Why? OF HERMETIAN $\hat{O}$'s

1) Eigenvalues of Eigen functions/vectors = REAL } good! things we measure are real

2) Eigenfunctions form a Complete basis in that function or vector space

See Griffiths or Townsend

↳ Key! You are guaranteed that solns to your problems can be built from these!

ex: in V.S.,

$\vec{F} = F_x \hat{i} + F_y \hat{j}$

Basis

or

F.S.,

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \left(\cos(n\omega_x x) \right) + b_n \left(\sin(n\omega_x x) \right) \right]$$

So: Always work w/ Eigenvalues & functions & vectors w/ Linear $\hat{O}$'s! Hermetian

So what are eigenfunctions (vectors) & values?

Q.M

Heisenberg, Dirac, Born Jordan

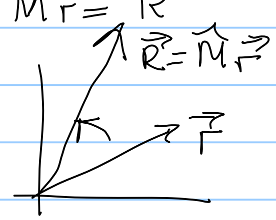
Lives V.S. (matrix)

Schrödinger

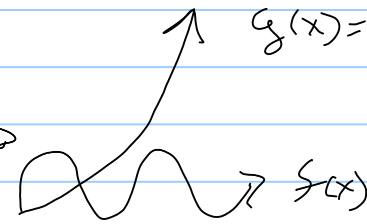
Born

Lives in F.S.

$\hat{M}\vec{r} = \vec{R}$



$\hat{O}$ s in general



$\hat{O}(f) = g(x)$

so $\hat{M}$ = rotation + stretch (ie. $\underbrace{M}$ = matrix)

Now there may be some vectors or functions that are special

$\hat{M}\vec{r} = c\,\vec{r}$ and $\hat{O}(f(x)) = c\,f(x)$

ie that the $\hat{O}$'s did not change the vector dir or the funct form

But all it did was change the length or multiplicative constant

In general real or complex.

So there might be

1.) V.S. $\hat{M}\vec{r} = c\vec{r}$

2.) F.S. $\hat{O}\, f(x) = c\, f(x)$

so $\vec{r}$ = eigen vector

$f(x)$ = eigen function

& c = eigen vector

Our goal will be to find Eigenvalues & Eigenvectors of Hermetian Linear $\hat{O}$'s because they are Physical!

V.S.

$\hat{M}\vec{r} = c\vec{r}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{pmatrix} r_x \\ r_y \end{pmatrix} = c\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

F.S.

$$\frac{d}{dx}\left(e^{mx}\right) = me^{mx}$$

ex: 2.5 Scherrer: Eigenfuncts & values of $\hat{D} \equiv \frac{d}{dx}$

Find eigenfunctions of $\hat{D}$

Looking for

$$\hat{D}\left[g(x)\right] = c\,(g(x))$$

or

$$\frac{d}{dx}(g(x)) = c\,(g(x))$$

Can separate variables → g(x) = entire variable

$$\frac{d(g(x))}{g(x)} = c\,dx$$

$$\int \frac{dg(x)}{g(x)} = \int c\,dx$$

* indef integral so need to keeps integration constant

$$\ln g(x) = cx + c_2$$

$$e^{cx+c_2} = g(x)$$

$$g(x) = e^{cx}\underbrace{e^{c_2}}_{\text{constant call it } A} = Ae^{cx}$$

There are an ∞ # of solutions of this form.

So: all functions of form Ae^{cx} = eigenfunctions of $\hat{D}$

∞ # of E.F. & E.V.

* Seems trivial, but in fact not all $f(x)$'s are eigenfunctions

ex:

$$\hat{D}(\sin(x)) = \cos(x) \neq c\sin(x)$$
$$\hat{D}[x^2] = 2x \neq cx^2$$
$$\hat{D}[\ln(x)] = \frac{1}{x} \neq c\ln(x)$$

$\therefore$ Eigen functions ARE special!

again QM Based on EF & EV of Hermetian Linear $\hat{O}$'s

Ex: 2.6 in Scherrer: $\underbrace{\text{Find EF \& EV of parity}}_{}$ $\hat{O} \equiv \hat{\Pi}$

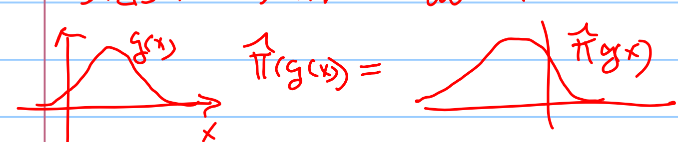
HUGE! in Fund Physics & Search for e^- EDM

Violations of $\hat{\Pi}$ come only as a result of weak nuclear force & make to distinguish is physics is being seen in a mirror or not!

idea: $\hat{\Pi}[g(x)] = \underbrace{g(-x)}$

this is the operation

$\hat{\Pi}$ reflects the function about $x=0$



So looking for

$$\hat{\Pi}[g(x)] = \underbrace{g(-x) = c\,g(x)}$$

don't know how to solve this easily so TRICK!

try $\hat{\Pi}^2[g(x)] = \hat{\Pi}\{\hat{\Pi}(g(x))\} = \hat{\Pi}(c\,g(x))$

$$\underbrace{\hat{\Pi}(g(-x))} = c\,\hat{\Pi}\,g(x)$$

$$g(x) = c^2 g(x)$$

Note: even when an EF of a Linear $\hat{O}$ is mult by an arbitrary constant, it remains an EF w/ the same EV

Now can solve this

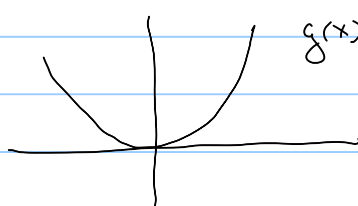
$$c^2 = 1 \quad \text{or} \quad c = \pm 1 \Big\} \text{ 2 Discrete eigenvalues}$$

Now the eigen functions:

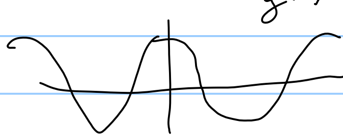
for $c_1 = 1$ = all even functions

$g(x) = g(x)$

$g(x) = x^2$



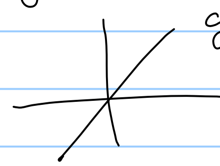
$g(x) = \cos(x)$



$c_2 = -1$ = all odd functions.

$g(x) = -1\,g(x)$

$g(x) = x$



$g(x) = \sin(x)$