Not the 'sag all' lecture for complex #s.

But perhaps handy --- shows usefulness.

You've got + #s --- i.e. 7 cows

\[ \text{add them} \]

3) Then subtract them need to invent (-) #s.

4) Then multiply --- no new inventions.

5) Then divide big #s \[ \frac{\#}{\#} = \text{rational} \# \]

Invent decimal equivalents

Signature = repeating decimal

5) Then say, \( 5 \times 5 = 25 \), so \( 25^{\frac{1}{2}} = \pm 5 \) ok

But then run into trouble w/l \( 2^{\frac{1}{2}} \)

\( (2)^{\frac{1}{2}} = ? \) Nothing works (see rational #5s).

So invent \( \sqrt{2} = \text{the right} \# \)

Such that

\[ (\sqrt{2})^2 = 2 \]

\( \sqrt{2} = \text{new irrational} \# \)

Signature = Never repeating decimal (ie it)
So everything is cool.

But can access all + it's from \( \sqrt{ } \)
But can't get to \(-\) it's from this operation.

So invent new thing why not \( \sqrt{2} \) non-repeating decimal

why not \( i \) such that \( i^2 = -1 \)

\[ i \cdot i = -1 \]

Is this useful? You betcha!

Here's what Feynman calls The Jewel of Mathematics!

Basic idea: Every algebraic entity has a geometric representation.

\[ 3 \leftrightarrow 0 0 0 0 \]

\[ 3x+1 \leftrightarrow \text{a line } y = mx + b \]

\[ 9 + 16 = 25 \] a circle

\[ \frac{1}{z} \text{ so on} \]
The geometrically is use $\cos \theta$ -- see nature wrap down with.

Ok, so now we've got $i$, such that $i^2 = 1$ is some imaginary $i^x$.

Well $e$ why not -- does it have geometric reps?

How can you see what $e^{ix}$ is made of? You got it. Taylor series.

$$e^x = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots$$

For small $x$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{-ix^2}{7!}$$

$$e^{ix} = (1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots) + i \left( x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \right)$$

Recall

Maclaurin Series expansion:

$$\cos \theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots$$

Then $$\sin \theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots$$

Then $e^{i\theta} = \cos \theta + i \sin \theta$. Woohoo!
New area complex #s have geometrical interpretation

\[ e^{i\theta} = \cos \theta + i \sin \theta = x + iy = z \]

so \[ e^{i\theta} = \text{A VECTOR!} \]

is \[ \theta = \phi(t) \]

\[ e^{i\phi(t)} \]

\[ \text{Convention from Feynman!} \]
\[ \hat{A} = \text{vector} \]
\[ \vec{A} = \text{matrix} \]
\[ A = \text{complex #} \]

Clock!

See Feynman
“\text{The Strange Theory of Light & Matter}”

Very Cool!

1) Having geometric sense is key

2) Having \( z \) is fun too. Often you must guess \( \theta \) relates to problems & then see them to work (DiSgy - Q's)

Here, is guess @ real # only might nor work
But start guessing w/ most general encompassing object that can be a vector \( x + iy \) or

scalar, \( y = 0 \)

You are much better off.
Because \( \mathbf{z} = \mathbf{x} + i\mathbf{y} \) is a complex vector

\[
\mathbf{z} = (x + iy, y - ix)
\]

Real and imaginary parts are linearly independent. Shame each other \( \mathbf{A} - \mathbf{CB} \neq \mathbf{0} \) for all \( \mathbf{C} \)

\[
(x - 2y) - \frac{1}{2} (2x - 4y) = 0
\]

\[
\frac{x}{y} = \frac{1}{2}
\]

or

\[
\begin{pmatrix}
1 & -2 \\
2 & -4
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= \begin{pmatrix}
2 \\
-4
\end{pmatrix}

\begin{pmatrix}
2 & -4 \\
0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 \\
0 & 0
\end{pmatrix}
\]

Echelon form

\[
\mathbf{r}_n = 1 \neq 0
\]

\( \Rightarrow \mathbf{z} \neq \text{linearly independent} \)

So being linearly independent you can do cool tricks

\[
\text{ex: } \mathbf{A} \text{ apples } + \mathbf{B} \text{ bananas} = \mathbf{X} \text{ apples } + \mathbf{Y} \text{ bananas}
\]

\[
\text{app & ban are linearly independent}
\]

So it is TRUE that

\[
\mathbf{X} = \mathbf{A}
\]

\[
\frac{1}{2} \mathbf{Y} = \mathbf{B}
\]

So, if you have a tough problem

\[
\text{lots of } \sin & \cos
\]
ie \[ \frac{\theta}{2} = \frac{3}{2} \text{ so am} \ldots \]

Then let all your variable go complex!

elevate them to the complex plane.

ex: if you have \( \cos \theta \rightarrow \text{Re}(e^{i\theta}) \)

Next 2 props!
Now we have to have a complex algebra break.

Say you have \( s_1 = \cos(\cdot 2\pi + x) \)

\[ s_2 = \cos(\cdot 2\pi + y) \]

\[ s_3 = s_1 + s_2 \] gives you'd have to do lots of trig identities.

Here's the idea:

elevate \( s \)'s \( \rightarrow \) complex (you've added imaginary part)

But that's okay, it's linearly indep.

Recall our 3 notation

\( \mathbb{R} \) or 3 algebras

\( \mathbb{D} \) vector

\( \mathbb{A} \mathbb{B} 's \) product

real 1 \& 2 = matrix algebra

\( \mathbb{A} 's \) tensor

\( \mathbb{C} \) complex

\( \mathbb{W} = \mathbb{A} \mathbb{B} \)
\[(\cos x + i\sin x) + (\cos y + i\sin y)\]

\[(\cos x + \cos y) + i(\sin x + \sin y)\]

\[S = \tan \left(\frac{\sin x + i\sin y}{\cos x + \cos y}\right)\]

\[S_T = \left|\frac{\tilde{s}}{s}\right| = \sqrt{\left(\frac{(\cos x + \cos y)^2 + (\sin x + \sin y)^2}{\cos x + \cos y}\right)} = \frac{s_T}{s_T} \frac{s_T}{s_T}\]

\[S_3 = s_1 + s_2 = s_T e^{i\cdot2\pi}\]

\[= s_T e^{i\cdot\pi} e^{i\cdot\pi} = s_T e^{i\cdot\pi}\]

\[S_3 = s_T \left[\cos(\cdot2\pi + \pi) + i\sin(\cdot2\pi + \pi)\right]\]

Now wanted \(s_3 = s_1 + s_2\) so take real parts

\[s_3 = \text{Re}(\tilde{s}_3) = s_T \cos(\cdot2\pi + \pi)\]

But other part

Swam linearly

Induced complex

part is also soln.

Since you got 2-1

Rooted it to 2 solns

You expect 2 solns.
So, when you see messes of \(\sin \) and \(\cos\), think making your life easier by elevating to complex plane.

In this case, complex \#s are simply a bookkeeping tool to keep independent solutions apart from each other.

\[ \text{Complex } \# \text{s arise because vectors are } \text{solns!} \]

But again vectors are linear independence solutions, so kind of same thing!
Multiplication of complex #s

ex: $(2+3i)(1-i) = 2 - 2i + 3i - 3i^2 = -1 + i$

but we now need a quick to

\[ \mathbf{A} = 2 + 3i = \begin{pmatrix} 1 \sqrt{13} \\ 2 \end{pmatrix} \]

\[ |A| = \sqrt{1^2 + 3^2} = \sqrt{13} \]

in Q.M. we will want $|\psi|^2 = \text{prob: } \langle \psi | \psi \rangle = |\psi|^2$

Bra-Ket notation of "inner product" or dot product

Bohm & Jordan + Dirac

\[ \text{recognized} \rightarrow \text{This form} \rightarrow \text{Heisenberg} \]

\[ = (A_x A_y) (A_x) = A_x^2 + A_y^2 \]

\[ \text{matrix Multiplication} \]

\[ \frac{1}{2} \text{ in SR, we had 4-vector dot prod: } x^a = (c, \vec{x}) \]

\[ 1 \rightarrow A_{\mu} x^\mu = (ct^2 / \sqrt{x^2}) \]

\[ \text{Gauss Einsten equation} \]

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]

\[ \text{Klein-Gordon} \]

\[ \text{Flote} \]

\[ \text{space metric} \]
So we have history of making things, rules, up here.

\[ |z|^2 = (x+iy)(x+iy) \]

Let's see:

\[ x^2 + ixy + ixy - y^2 \]

\[ = x^2 - y^2 + 2ixy \]

Nope?

Then \[ |z|^2 = z^* \cdot z \text{ where } * \equiv \text{complex conjugate} \]

rule where over

\[ +i \rightarrow -i \]

\[ -i \rightarrow +i \]

So here:

\[ z^* \cdot z = (x-iy)(x+iy) \]

\[ = x^2 + xixy - xiyy - i^2 y^2 = x^2 + y^2 \]

That's it... if you have a complex number, the magnitude of complex number is

\[ |z|^2 = \sqrt{z \cdot z^*} \]

\( \cos \theta \text{ so } | \hat{A} | = \sqrt{x^2 + y^2} \)
But back to our sample problem

\[ \hat{A} \hat{B} = 2 \quad \hat{A} = 3 + 3i \]
\[ \hat{B} = 1 - i \]

\[ \hat{A} = \begin{bmatrix} 3 & 3i \\ -3 & 1 \end{bmatrix} \]

\[ |\hat{A}| = \sqrt{\hat{A}^* \hat{A}} = \sqrt{(2 - 3i)(2 + 3i)} \]
\[ = \sqrt{4 + 9} \]
\[ = \sqrt{13} \]

\[ |\hat{A}| = \sqrt{13} \]

\[ \hat{A}^\circ \]
\[ \hat{A} = \begin{bmatrix} \sqrt{13} & 3i \\ -3 & -i \end{bmatrix} \]
\[ \sigma = \tan^{-1} \frac{3}{2} \]

\[ \hat{A} = |\hat{A}| e^{i\sigma} \] because \[ |\hat{A}| \cos \sigma + i |\hat{A}| \sin \sigma \]
\[ = 2 + i3 \]

\[ \hat{A} \hat{B} = (3 + 3i)(1 - i) = -1 + i \]

\[ \hat{A} \hat{B} = |\hat{A}| e^{i\sigma} |\hat{B}| e^{i\sigma} \]
\[ = \sqrt{13} e^{i\tan^{-1} \frac{3}{2}} \cdot \sqrt{2} e^{i\tan^{-1} \frac{1}{2}} \]
Got a bit messy here, but back up to early pt

complex \# = geometric vector

so

\[ Re = R \cos \omega + i R \sin \omega \]

way easier
to do math

multi divid

Calculus (dissimilar)

Integrate

w/ exponentials than
w/ vectors!
Another Big Point ----

we know that \( x + iy = \text{vector} \)

But what is \( \frac{a + ib}{c + id} \) as close.

\[ \frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{c^2 + d^2} \]

\[ = \frac{ac + ib(c - da) + bd}{c^2 + d^2} = \frac{1}{c^2 + d^2} (x + iy) \]

const

good vector again

Svom