

#5

Not the 'say all' lecture for
Complex #s

But

perhaps handy ---- shows usefulness.

H.W. all
examples
in this
sect
+
2.1
2.3
2.4
2.6

you've got + #s ---- ie 7 cows

1)

add them

2) then subtract them need to invent (-) #s

3) then multiply - no new inventions

↑ rules

4)

then divide by #s

$$\frac{\#}{\#} = \text{rational } \#$$

invariant decimal
equivalents

Signature = repeating
decimal

5) then say, $5 \times 5 = 25$, so $25^{1/2} = \pm 5$ OK

But then run into trouble w/ $2^{1/2}$

$(2)^{1/2} = ?$ Nothing works (ie Rational #s)

so invent $\sqrt{2}$ = the right #

such that

$$(\sqrt{2})^2 = 2$$

$\sqrt{2}$ = new irrational #

Signature Never repeating
decimal (ie π)

So everything is cool

But can access all + #'s from $()^2$
But can't get to (-) #'s from this operation.

So invent new thing, why not $\sqrt{2} = \infty$ non-repeating decimal

why not i , such that $i^2 = -1$

$$\therefore \sqrt{-1} = i$$

Is this useful? Yoo betcha!
why?

Here is what Feynman calls the Jewel of mathematics!

Basic idea: Every algebraic entity has a geometric representation.

$$3 \Leftrightarrow \textcircled{1} \textcircled{1} \textcircled{1}$$

$$3x + 1 \Leftrightarrow \text{a line } mx + b$$

$$9 + 16 = 25 \quad \text{a circle}$$

⋮ so on

i^2 s, The geometric, is use \sin -- see nature
Wbp down math.

Ok, so now we've got i , such that $i^2 = -1$
is some imaginary #

well e^{ix} why not -- does it have geometric
rep?

How can you see what $f(x)$ is made of?
You get it Taylor series

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots$$

for all x

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + \dots$$

$$e^{ix} = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right) + i \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)$$

recall

Maclaurin series expansion

$$\cos \theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \dots$$

$$\sin \theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \dots$$

Then

$$e^{ix} = \cos \theta + i \sin \theta$$

Wow!

New even complex #'s have geometrical interpretation

$$e^{i\theta} = \cos\theta + i\sin\theta = \begin{array}{c} \text{Imag} \\ \nearrow \\ \theta \\ \searrow \\ \text{Real} \end{array} = x + iy = z$$

so $e^{i\theta} = \underline{\underline{A \text{ VECTOR!}}}$

is $\theta = \theta(t)$

$$e^{i\theta} = \begin{array}{c} \nearrow \\ \theta(t) \\ \searrow \end{array}$$

Convention from
Griffiths & Me!

\vec{A} = vector

A_j = matrix

\tilde{A} = complex #

Clock!

See Feynman

"The Strange

Theory of

Light & Matter"

Very Cool!

1) Having geometric feel is key

2) Having z is fun too. Often you must guess @ solutions to problems & then see them to work (Disq-Q's)

Here, is guess @ real # only, might never work
But is start guessing w/ most general 'encompassing'
object that can be a vector $x + iy$ or
scalar, $y = 0$

You are much better off

Because $z = x + iy$ is like a vector

↑ ↑
Real imaginary parts are
linearly independent from each
other

$$* \vec{A} - c\vec{B} \neq 0 \text{ for all } c$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$$



$$\text{or } \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \xrightarrow{2 \times 1} \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$$

$$2 - 1 \begin{pmatrix} 2 & -4 \\ 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}}$$

$$= \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

echelon row

for $m = 1 \Rightarrow$

$\therefore \vec{A} \not\perp \vec{B} \neq$ linearly indep.

↳ being linearly independent you can do
cool tricks

ex: A apples + B bananas = X apples + Y bananas

app & ban are linearly indep

\therefore it is true that

$$X = A$$

$$\frac{1}{2} Y = B$$

↳ so, if you have a tough problem
w/ lots of sines & cos

ie mult $\frac{1}{s} \frac{1}{s} \frac{1}{s}$ so on ...

then let all your variables go } elevate
complex! } them to
the
(complex plane

see
electronics!

ex: if you have $\cos t \rightarrow \text{Re}(e^{i t})$
then

EMI
did it
Griffiths!

Next
2 p apps!

Plus page & next from P#458, E&M2 g'nissch
 chpt 9, waves 2 lect

Now we have to have a complex algebra break.

Say you have $S_1 = \cos(.7\pi + x)$

$S_2 = \cos(.7\pi + y)$

$S_3 = S_1 + S_2 = ?$ yikes you'd have to do lots of trig identities

Here's the idea:

elevate S 's $\rightarrow \tilde{S}$ complex (you've added imaginary part
 But that's OK, it's linearly indep)

so $S_1 \rightarrow \tilde{S}_1 = e^{i(.7\pi + x)}$

$S_2 \rightarrow \tilde{S}_2 = e^{i(.7\pi + y)}$

Then $\tilde{S}_1 = e^{ix} e^{i.7\pi}$
 $\tilde{S}_2 = e^{iy} e^{i.7\pi}$

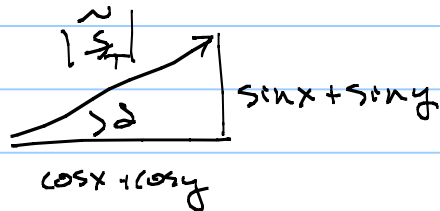
$\tilde{S}_3 = \tilde{S}_1 + \tilde{S}_2 = (e^{ix} + e^{iy}) e^{i.7\pi}$
 $= \underbrace{\tilde{S}_3}_{\tilde{S}_3} e^{i.7\pi}$

recall our 3 notation for 3 algebras

- 1) Vector $\vec{A} \cdot \vec{B}$
 $\vec{A} \otimes \vec{B} \rightarrow \vec{A} \times \vec{B}$
 real 1 & 2 = matrix algebra
- 2) Tensor $\vec{r} = \underline{A} \vec{r}$
- 3) complex $\tilde{C} = \tilde{A} \tilde{B}$

$$(\cos x + i \sin x) + (\cos y + i \sin y)$$

$$(\cos x + \cos y) + i(\sin x + \sin y)$$



$$S = \tan^{-1} \left(\frac{\sin x + \sin y}{\cos x + \cos y} \right)$$

$$S_T = |\tilde{S}_T| = \sqrt{(\cos x + \cos y)^2 + (\sin x + \sin y)^2} = \sqrt{S_T^* S_T}$$

$$\tilde{S}_3 = \tilde{S}_1 + \tilde{S}_2 = \tilde{S}_T e^{i \cdot 7\pi}$$

$$= S_T e^{i\alpha} e^{i \cdot 7\pi} = S_T e^{i(7\pi + \alpha)}$$

$$\tilde{S}_3 = S_T [\cos(7\pi + \alpha) + i \sin(7\pi + \alpha)]$$

Now wanted $S_3 = S_1 + S_2$ so take real parts

$$S_3 = \text{Re}(\tilde{S}_3) = S_T \cos(7\pi + \alpha)$$

\Rightarrow But other part, from linearly indep complex part is also soln.

? Since you extracted it to 2-D you expect 2 solns

So, ① when you see messes w/ \sin & \cos ,
 think making your life
 easier by \rightarrow elevating to complex plane

\updownarrow
 doing your work

answer

\leftarrow
 Bring back to Real
 answers using
 linearly indep prop.

Note: Bonus is you might
 get 2 good answers!

In this case
 Complex #'s are
 simply a bookkeeping
 tool to keep
 independ solns apart
 from each other

② or have messy d'fy - Q that
 you don't know the answer to,
 guess

$$\frac{dy}{dx} = \tau y \quad y(x) = e^{px} \quad \text{if } p = \text{complex}$$

$$\text{Then } y(x) = e^{ipx} = \underbrace{(\cos px + i \sin px)}_{\text{vector!}}$$

Have complex
 #'s arise because
 vectors are
 solns!

But again vectors
 are linearly indep
 solns, so
 kind of same
 thing!

But this is Big
 idea.

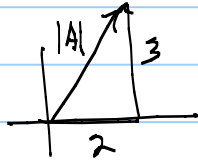
Multiplication of complex #'s

Complex Conjugating!

ex: $(\vec{A})(\vec{B}) = 2 + -2i + 3i - 3 = -1 + i$

or:

$\vec{A} = 2 + 3i =$



clear that $|A| = \sqrt{2^2 + 3^2} = \sqrt{13}$

but we now need a trick to find complex # magnitudes

don't worry, we have history of this...

1) $|\vec{A}|^2 = \vec{A} \cdot \vec{A} = A_x^2 + A_y^2$ "dot" rule

2) in Q.M. we will want $|\psi|^2 = \text{prob: } \langle \psi | \psi \rangle = |\psi|^2$
 Bra-ket notation of "inner product" or dot product

Born & Jordan + Dirac recognized this as Gram-Hermitian.

$= (A_x \ A_y) \begin{pmatrix} A_x \\ A_y \end{pmatrix} = A_x^2 + A_y^2$

= matrix multiplication!

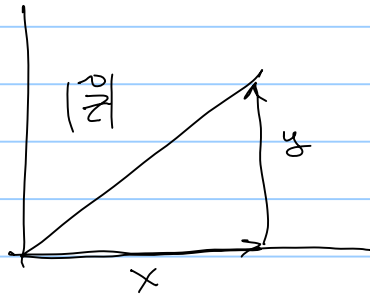
3) in SR we had 4-vector dot prod: $X^\mu = (ct, \vec{X})$
 then $X_\mu = g_{\mu\nu} X^\nu$ ($g_{\mu\nu}$ = metric tensor from Einstein equation)

so $I^2 = X_\mu X^\mu = (ct^2 - |\vec{X}|^2)$

$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ Minkowski Flat-space metric

So -- we have history of walking things, rules, up.

here



$$|z|^2 = (x+iy)(x+iy)$$

lets see

$$x^2 + ixy + ixy - y^2$$

$$= x^2 - y^2 + 2ixy$$

Nope?

Try $|z|^2 = z^* z$ where $*$ \equiv "taking the complex conjugate"

rule where ever

$$+i \rightarrow -i$$

$$-i \rightarrow +i$$

so here

$$z^* z = (x-iy)(x+iy)$$

$$= x^2 + xiy - xiy - iy^2 = x^2 + y^2$$

That's it -- if you have a complex z , then mag of complex z is

$$|z| = \sqrt{z^* z}$$

$$\text{just as } |\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$$

But back to our simpler problem

$$\begin{aligned} \vec{A} \vec{B} &= ? & \vec{A} &= 2+3i \\ & & \vec{B} &= 1-i \end{aligned}$$

$$\vec{A} = \begin{array}{c} \begin{array}{c} |A| \\ \nearrow \\ 2 \end{array} \quad \begin{array}{c} \nearrow \\ 3 \end{array} \\ \end{array} \quad \text{now } |A| = \sqrt{A^*A} = \sqrt{(2-3i)(2+3i)}$$
$$= \sqrt{4+9}$$
$$= \sqrt{13}$$

$$|A| = \sqrt{13}$$

so

$$\vec{A} = \begin{array}{c} \begin{array}{c} \sqrt{13} = |A| \\ \nearrow \\ 2 \end{array} \quad \begin{array}{c} \nearrow \\ 3 \end{array} \\ \end{array} \quad \sigma = \tan^{-1} \frac{3}{2}$$

$$\begin{aligned} \vec{A} &= |A| e^{i\sigma} \quad \text{because } = |A| \cos \sigma + i |A| \sin \sigma \\ &= 2 + i3 \end{aligned}$$

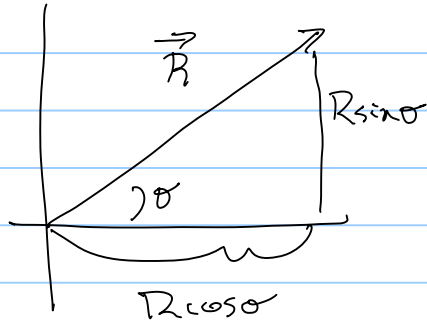
$$\therefore \vec{A} \vec{B} = (2+3i)(1-i) = -1 + i$$

$$\begin{aligned} \text{or } &= |A| e^{i\sigma_A} |B| e^{i\sigma_B} \\ &= \sqrt{13} e^{i \tan^{-1} \frac{3}{2}} \sqrt{2} e^{i \tan^{-1} \frac{-1}{1}} \end{aligned}$$

Got a bit messy there, but back up to early pt

Complex # = geometric vector

So



$$Re^{i\phi} = R \cos \phi + i R \sin \phi$$

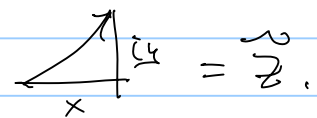
way easier
to do math

mult, div &

Calculus (differentiate & Integrate)

w/ exponentials than
w/ vectors!

Another Big Point ----

We know that $x + iy = \text{vector}$  $= z$.

But what is

$\frac{a+ib}{c+id}$? no clue

so
always do this

$$\frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{(a+ib)(c-id)}{c^2+d^2}$$

$$= \frac{ac + i(bc - da) + bd}{c^2 + d^2} = \frac{1}{c} (x + iy)$$

↑
const

↑
good
vector
again
from