E.F. Deveney / BSC Physics PH 402 Quantum I

Sherrer C8

Electron Spin

Note Title 5/3/2006

Previous H-atom

$$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$$\downarrow$$

$$\hat{H}\psi_{n,r,\theta,\phi} = E_n\psi_{n,r,\theta,\phi}$$

after write m_ℓ

Get $$\Psi(t,r,\theta) = \underbrace{e^{-i\frac{E_n t}{\hbar}} R_{n\ell}(r) Y_\ell^{m_\ell}(\theta,\phi)}_{\text{wavefunction of state}} = \underbrace{|n\ell m_\ell\rangle}_{\text{abstract state ket}}$$

4-linearly indep dimensions

3-"Spacial" degree's of freedom

need 3 #'s to describe e^- in space $n\ell m_\ell$

$$E_n = \frac{-13.6\,eV}{n^2}$$; $n = 1, 2, 3\ldots$ = prime Quantum #

ℓ = orbital angular momentum

ℓ = "shape" of orbital

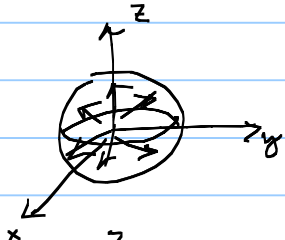
$\ell = 0 = s$

$\ell = 1 = p$

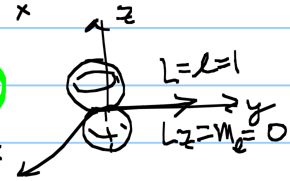
$\ell = 2 = d$

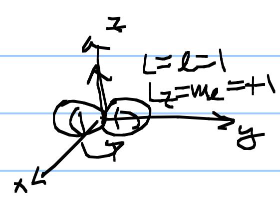
$\ell = 3 = f$

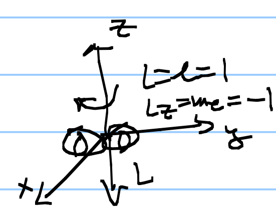
m_ℓ = orientation of orbital



$\ell = 0 \equiv$ "s" orbital $m_\ell = 0$







is that it? ie 3 space degrees of freedom to describe "STATE" of H-atom e^-?

Turns out NO, will need
Another Degree of freedom
to describe e^-
Linearly indep of SPACE

SPIN!

$$\Psi_{TOT} = e^{-i\frac{E_n t}{\hbar}} R_{n\ell}(r)\, Y_\ell^m(\theta,\phi) \underbrace{|\uparrow,\downarrow\rangle}_{}$$

new 2-D
spin space
indep of all
others

$$\Psi_{TOT} = |n, \ell, m_\ell, s, m_s\rangle$$

s & m_s behave exactly as

$\vec{\ell}$ & m_ℓ = Angular momentum!

CALLED
Intrinsic
Angular momentum,
you just have!

$$\begin{cases} s = \text{Spin angular momentum} \\ m_s = \text{proj of spin angular momentum} \end{cases}$$

Further:

Further: Spin ultimately determines a great deal what particles

Spin $= n\left(\frac{1}{2}\right)$

$n = 1, 2, 3 \ldots$

Fermions

Fermi-Dirac Statistics!

Pauli-Exclusion Princ

Spin $= n$

$n = 0, 1, 2, 3 \ldots$

Bosons

Bosé-Einstein Stats

Lasers!

BEC's!

Laser atoms!
even though
atoms made of
Fermions, they
add = Boson

n-particles

n-particles all in 1 state

Spin: 1922 Stern & Gerlach: space quantization Physics Today, Dec 03: 53-59

AJP 71, (11) 2003, 1103-1107
Direct Dem of the transverse
Stern - Gerlach effect

1925 Goudsmit & Uhlenbeck: Physics Today
June 1976, 40-48

its spin!

$$\vec{\mu}_s = \frac{g_s q_e}{2m_e} \vec{S}$$

$g_s = 2$: Dirac relativistic equat

2004: PRL B 592, 1, 2004, S. Eidelman et al.

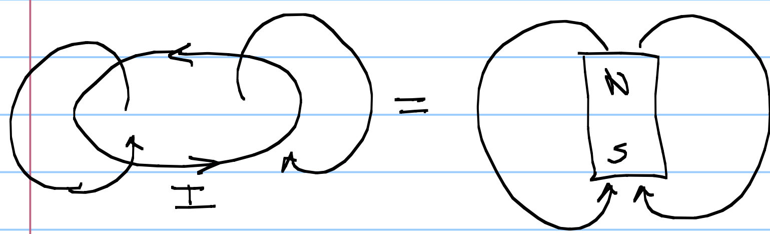
$g_s = 2.0023193043718 \pm .00000000000076$

Also

g-OLOGY: American Scientist
Vol 92 212-216
= Hundreds of
Feynman diagrams to get

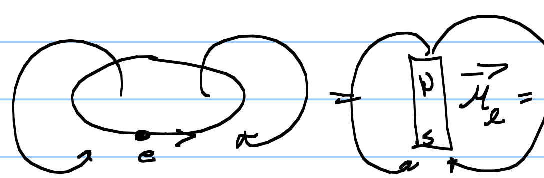
Spin: need to know about magnets!

I.) current loops make $\vec{B}$-fields = Magnets



In fact, macroscopic size magnets = sum of $\sim 10^{23}$ atomic sized magnets

atomic sized magnets = magnetic dipoles

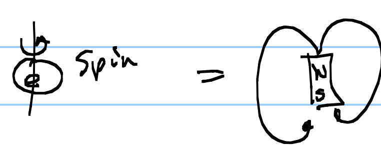


$\vec{\mu}_e = \vec{\mu}_{orbit} = I\vec{A}$

current

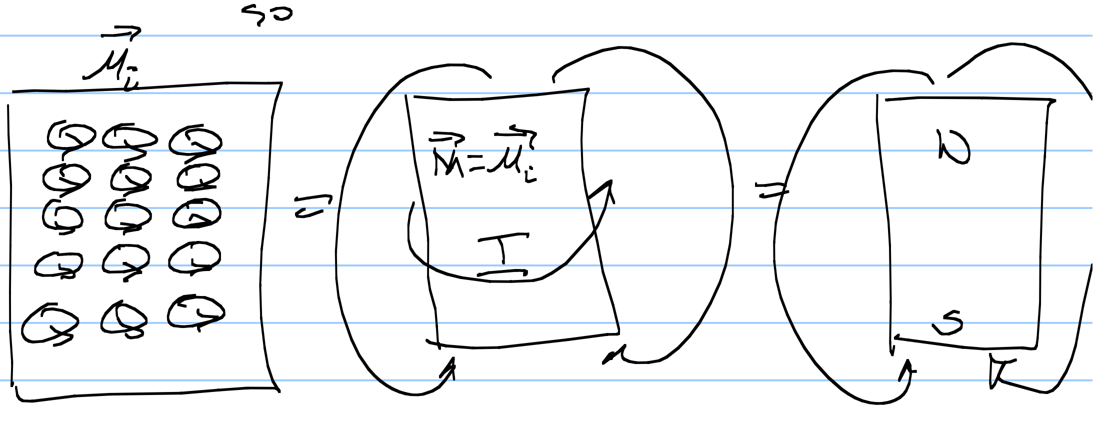
Area! see intro

Alright Admit Spin over here! $\Longrightarrow$



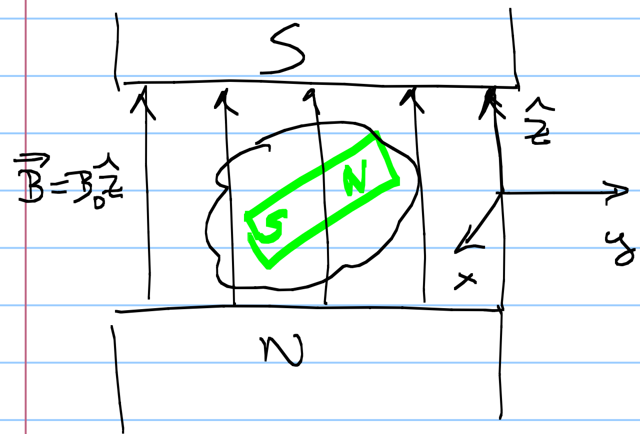
$\vec{\mu}_s = \vec{\mu}_{spin}$

so



We'll get back to the details but here is what we know from intro about magnets!

Magnets in Constant $\vec{B}_{external}$ fields



Constant $\vec{B}$ in $\hat{z}$ dir

$$\frac{\partial \vec{B}}{\partial z} = 0$$

$$\sum \vec{F}_{ext} = m\vec{a}$$

$$\big/ \;0\; \big| = m\vec{a}$$

$$\vec{a} = 0$$

No translation!

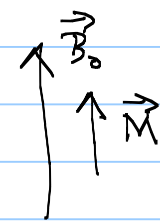
$$\sum \vec{\tau}_{ext} = I\vec{\alpha}$$

$$\big/ \;\vec{M}\times\vec{B}\; \big| = I\vec{\alpha}$$

$\therefore \alpha \neq 0$

& magnet rotates!

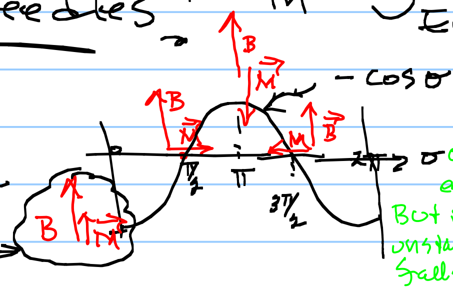
rotation, no translation in constant $\vec{B}_{ext}$ field. to



ex: Compass Needles! in $\vec{B}_{Earth}$

where: $E_p = -\vec{M}\cdot\vec{B} = -|M||B|\cos\theta$

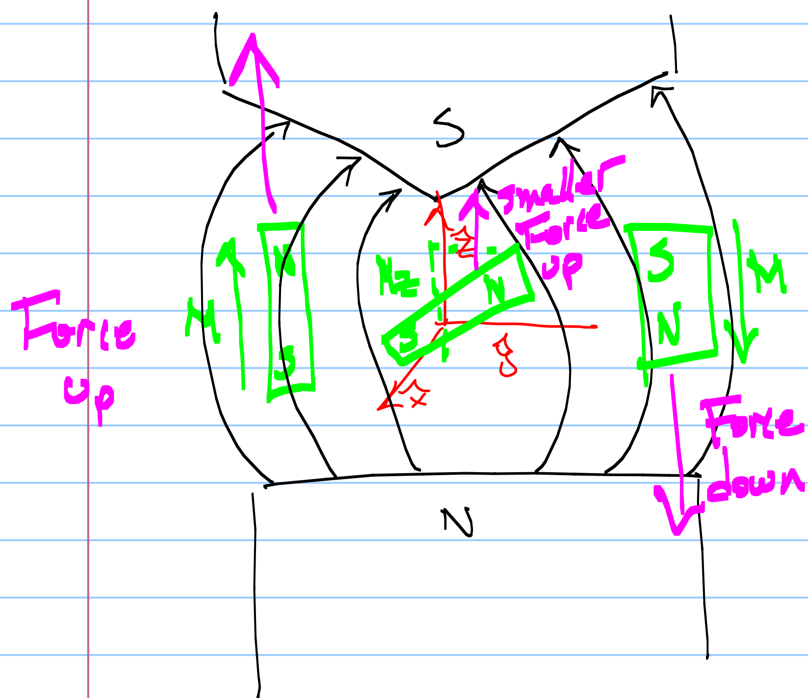
groundstate = Lowest E_p = aligned



Note: $\theta = \pi =$ equilib $\uparrow\vec{B}$ $\downarrow\vec{M}$

But it is unstable! quickly fall to $\uparrow\uparrow$

Magnet's in "spacially" varying $\vec{B}_{ext}$.



here:

$$\frac{\partial \vec{B}}{\partial z} \neq 0$$

HERE

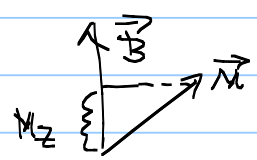
$$\sum F_{ext} = m\vec{a}$$

recall

$$\vec{F} = -\vec{\nabla} E_p = -\frac{\partial}{\partial z}(-\vec{M}\cdot\vec{B}) = +\frac{\partial}{\partial z}(M_z B_z) = \frac{\partial B_z}{\partial z} M_z = m a_z$$

$$F_z = \frac{\partial B_z}{\partial z} M_z$$

so

IF you have  & $\frac{\partial \vec{B}}{\partial z} \neq 0$ then

you get a Net FORCE in $\pm Z$ direction ∴

Magnet TRANSLATES in Z direction

$\propto$ to M_z = proj of mag on z-dir.

Big pt: $\vec{M}$'s in $\vec{B}$ where $\frac{\partial \vec{B}}{\partial z} \neq 0$
experience
Force and translate in z dir
proportional to proj of $\vec{M}$ in
z-axis!

Quick unit check

$$F_L = q\vec{v} \times \vec{B}$$

so $B = \frac{F}{C \frac{m}{s}} = \frac{N}{C} \frac{sec}{m} = \underline{\underline{\text{Gauss}}}$

pretty small!

Tesla $= 10^4$ Gauss

*Warning SI units = the Gauss

$$E_p = -\vec{M} \cdot \vec{B} = -\left(\sum_i \vec{\mu}_i\right) \cdot \vec{B}$$

magdipoles

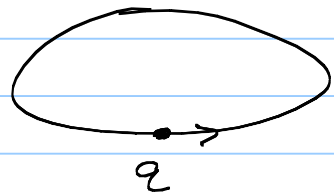
$= I A \cdot B$

$$\frac{\cancel{C}}{\cancel{s}} m^2 \frac{N}{\cancel{C}} \frac{\cancel{sec}}{m} = N \cdot m = J$$

OK!

Let's get back to atomic magnetic dipoles:

$\vec{\mu} = I\vec{A}$ consider, e^- in an orbit $\left(\ell = \begin{matrix} 0 & s \\ 1 & p \\ 2 & d \\ 1 & f \end{matrix}\right)$



or $I = \dfrac{\text{charge}}{\text{Time}} = \dfrac{q_e}{(\text{Time})}$

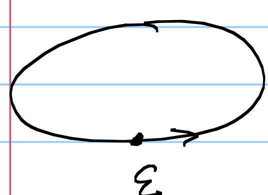
$(\text{Time}) = \dfrac{\text{distance}}{\text{veloc}} = \dfrac{2\pi r}{v}$

veloc? well $\vec{L} = mvr$

so $v = \dfrac{L}{mr}$

$$\therefore \quad I = \frac{q}{\dfrac{2\pi r}{\frac{L}{mr}}} = \frac{q}{\dfrac{2\pi r^2 m}{L}} = \frac{qL}{2\pi r^2 m}$$

so

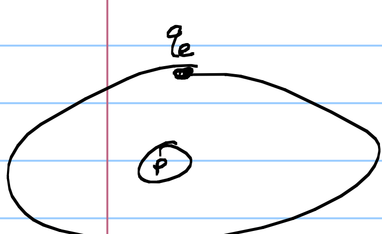


orbit $\vec{L}$

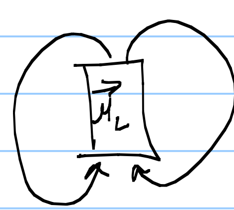
$$\vec{\mu}_L = I\vec{A} = \frac{q\vec{L}}{2\cancel{\pi r^2} m}\cancel{\pi r^2}$$

$$\vec{\mu}_L = \frac{q_e}{2m_e}\vec{L}$$

That's HUGE! H-atom orbit = Magnet!

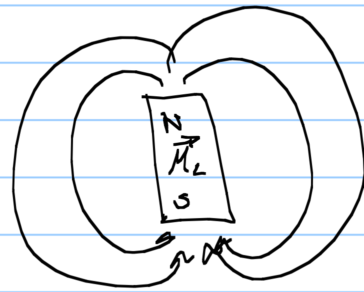


$\cong$



; $\vec{\mu} = \dfrac{q_e}{2m_e}\vec{L}$ ← orbital L

so

H atom $\cong$ 

$\vec{\mu}_L = \frac{q_e}{2m_e}\vec{L}$ since $\vec{L}$ has units of angular momentum

$\hbar$ = angular momentum = J·sec

So convenient to define

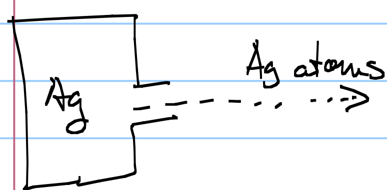
$\vec{\mu}_{Bohr} = \frac{e\hbar}{2m_e} = 9.3\times10^{-24}\, A\cdot m^2$ = Bohr Magneton

& in general $\vec{\mu} = \frac{\mu_B}{\hbar}\vec{L}$

Enter Stern & Gerlach: 1922

P.T. Dec 2003
pg 53–59
Must
READ

Historical
Human
Interest
wow!



Heat up

Now $^{47}Ag = \underbrace{\overbrace{[\text{Closed Shell}]}^{(46 e^-\text{'s})}}_{\substack{\text{Nobel} \\ \text{Gas}}} \underbrace{5s^1}_{\substack{\Uparrow \\ \text{lonely} \\ 5s\ e^-}}$ $\cong$

$= \vec{\mu}_{5s}$ of $5s^-$ ONLY!

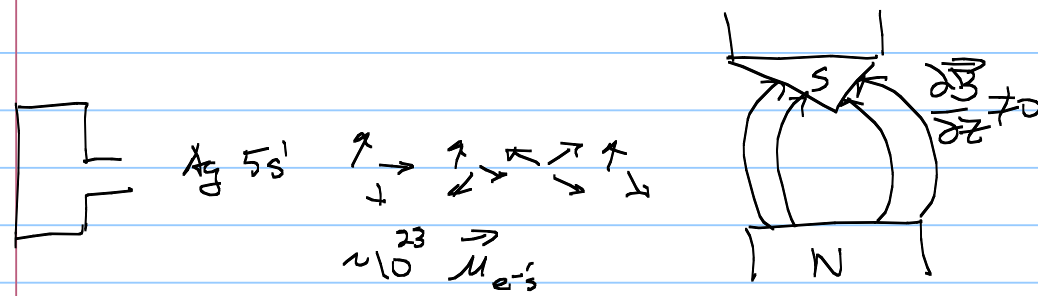
INERT
CHEMICALLY
So SUM of
all e^-'s do nothing

Conclude all 46 e^-'s
in orbits
= 46 magnets, $\vec{\mu}$'s
Add up to Zero
Exactly

So a Beam of Silver atoms $\cong$ Beam of
$5s\ e^-\ \vec{\mu}$'s

We can study Fund prop of e^-'s alone w/ This Technique

SG $\hat{z}$



now recall

in $\frac{\partial B}{\partial z} \neq 0$ = spacially invariant $\vec{B}$ field

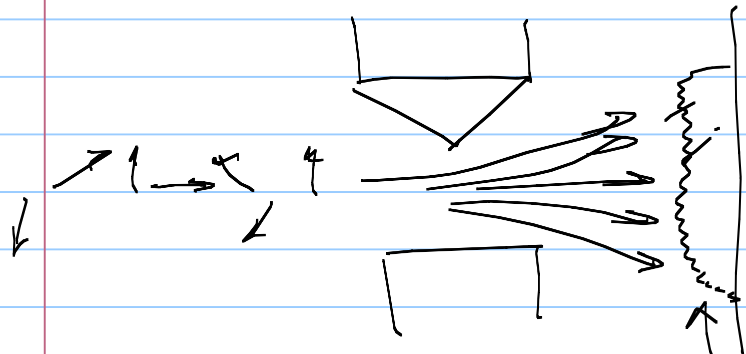
you get a Force on the tiny magnets where

$$\vec{F}_z = \frac{\partial B_z}{\partial z}\mu_z$$

So $\vec{\text{Force}}$ & thus Translation in z direction on each e^- $\vec{\mu} \propto \vec{\mu}_z$ $\left(\text{ie } \vec{a}_z = \frac{F_z}{m},\ z_f = z_i + v_i t + \frac{1}{2} a_z t^2\right)$

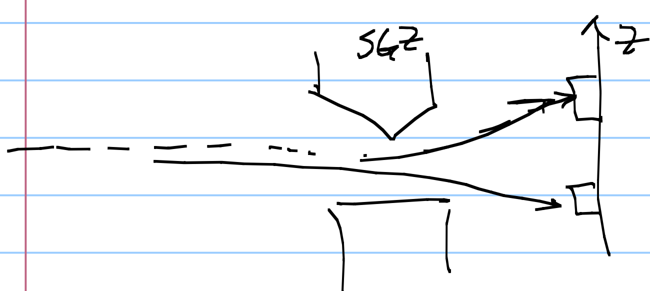
SINCE 10^{23} Randomly oriented atoms, $\vec{\mu}_z$ = randomly oriented

Now if 10^{23} truly randomly oriented then



We expect smeared out

But... here is what happens

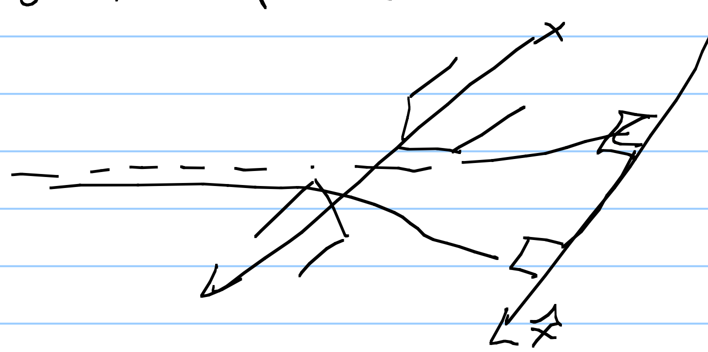


only ↑ or ↓

None in between

?

Try putting magnets this dir



Again!

2 only!

No matter the direction you always get only 2 Bloks of Atoms!

S-G: e^-'s Again are Quantized!

exist in jumps!

But here jumps are not in energy

Jumps, Quantization is in Space

* if you chose mag dir = $\hat{z}$

you get e^-'s either ↑ or ↓ in z dir.

* if you chose any dir of mag

you always only get ↑ or ↓ in That direction.

CONCLUDE: Once you define "Measure" by putting on B_{ext} field.

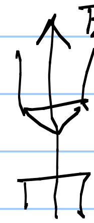
The e^- Becomes Either ↑ or ↓ w/ respect to Spacial dir of That field.

Copenhagen Interp!

As if

$\nearrow\!\uparrow\!\swarrow\!\swarrow \nearrow\!\uparrow = \underbrace{|\uparrow\downarrow\swarrow\nearrow\uparrow\rightarrow\rangle}_{\text{super posit of EVERYTHING}}$

$B\hat{z}$



$=$ ↑ or ↓ say $|\uparrow\rangle$ That's it

NEXT: Goodsmit & Ulhenbeck 1925

Read P.T. June 1976 pg 40–48

For Great Details!

Must READ

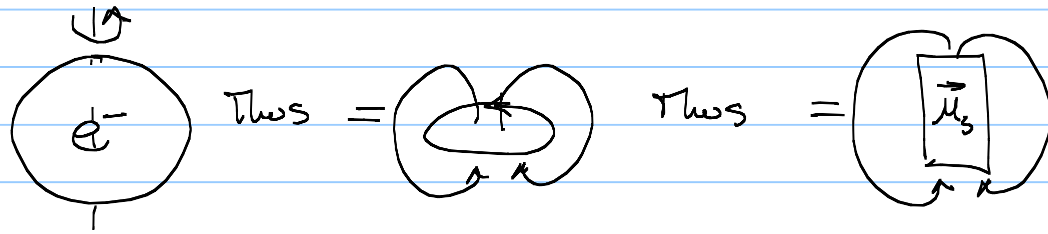
Historical

Everything!

Human interest!

Based on mostly Spectroscopic experiments

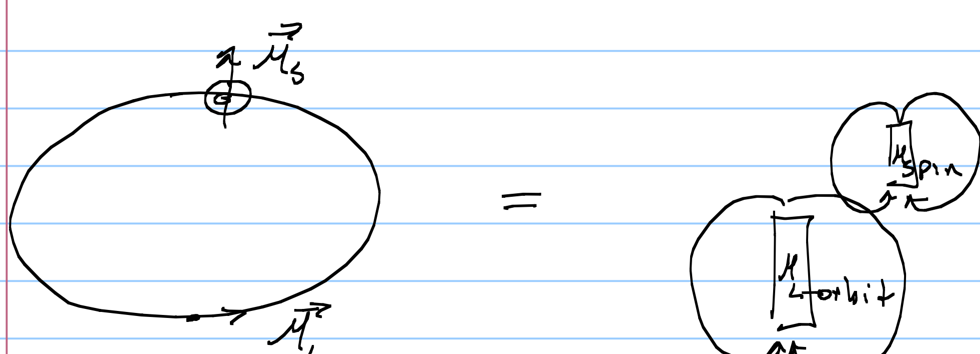
That e^- was doing the equivalent of "spinning"



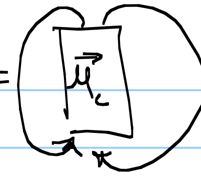
tiny mag dipole

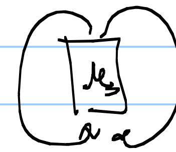
So

atom now Looks Like



= Tiny Magnets

Now: we previously computed $\vec{\mu}_L = \frac{q}{2m}\vec{L}$ = 
for
angular momentum.

Why not $\vec{\mu}_S = \frac{q_e}{2m_e}\vec{S}$ = 

Well thats WRONG! didn't get experiments exactly right... needed

$$\mu_S = (2)\frac{q}{2m}\vec{S}$$ to work?

mistake... wrong picture?

No... for relativity!

Using Dirac relativistic theory you get

$$\vec{\mu}_{S_{e^-}} = 2\frac{q_e}{2m_e}\vec{S}$$ EXACTLY!

* So no longer say $\vec{\mu}_L = \frac{q}{2m}\vec{L}$

say $\vec{\mu}_L = \frac{gq}{2m}\vec{L}$ g = gyromagnetic spin factor

But.... today we can do even more...

Dirac Theory $\Rightarrow$ Quantum Field Theory

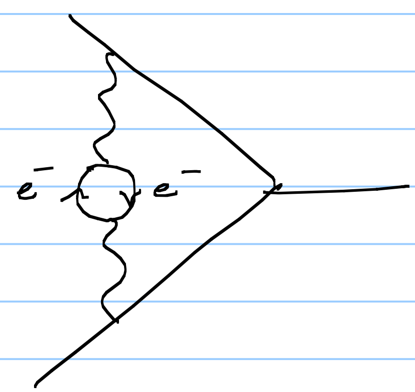
& can compute g_e even more precisely

See "g-ology" Brian Hayes Amer. Sci. #92, p 212

&

$g = 2.0023193043718 \pm .00000000000075$

using 100's of Feynman diagrams that include



vacuum $e^- + e^+$ pair production

& These #'s are confirmed experimentally

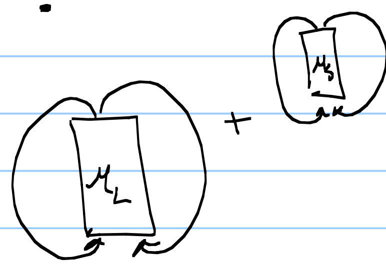
see Physics Letters B592, 1, 2004

If you have a new Theory, you must get this Better

(g_e)

ONE of most precise #'s known to man!

It all works!

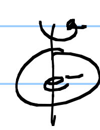
H-atom = 

where $\vec{\mu}_L = \frac{q_e}{2m_e}\vec{L}$

& $\vec{\mu}_s = \frac{g_s q_e}{2m_e}\vec{S}$

which means $\vec{S}$ acts just like a real angular momentum!

$\therefore$ $\vec{L}$ = angular moment

$\vec{S}$ = Intrinsic angular momentum!
Fundamental to e^- itself
not a
result of the 

picture
which breaks
down!

Now

Since we are looking for

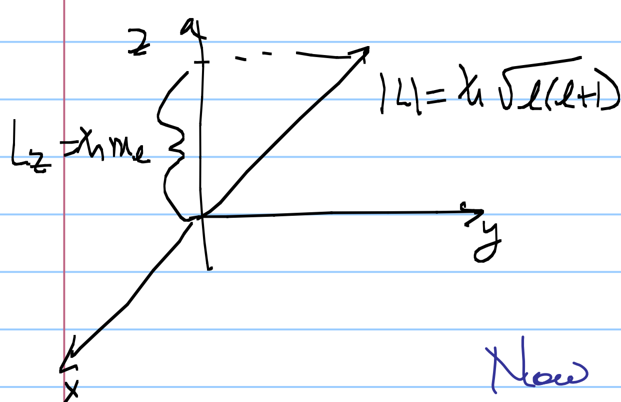
$$\Psi = R_{nl}(r)\, Y_l^{m_l}(\theta,\phi)\, |\text{spin}\rangle$$

we can use everything we know about $\vec{L}$ angular momentum

$$\left.\begin{aligned} [L_x, L_y] &= i\hbar L_z \\ [L_z, L_x] &= i\hbar L_y \\ [L_y, L_z] &= i\hbar L_x \end{aligned}\right\} \text{ No shared eigenstates}$$

But $[L^2, L_z] = 0$

so can get simultaneous eigenstates of L^2 & L_z



$$L^2 Y_l^{m_l} = \hbar^2 l(l+1) Y_l^{m}$$

or $L^2 |l\, m_l\rangle = \hbar^2 l(l+1) |l\, m_l\rangle$

$$L_z Y_l^{m_l} = \hbar m_l Y_l^{m}$$

$L_z |l\, m_l\rangle = \hbar m_l |l\, m_l\rangle$

* $l \le m_l \le -l$ integer steps

Now Here

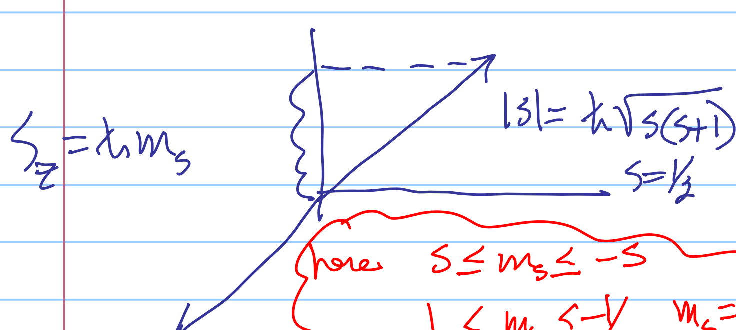
$$\left.\begin{aligned} [S_x, S_y] &= i\hbar S_z \\ [S_z, S_x] &= i\hbar S_y \\ [S_y, S_z] &= i\hbar S_x \end{aligned}\right\} \text{ no shared eigenstates}$$

But $[S^2, S_z] = 0$

so must be spin eigenstate of both S^2 & S_z

$$S^2 |s\, m_s\rangle = \hbar^2 s(s+1) |s\, m_s\rangle$$

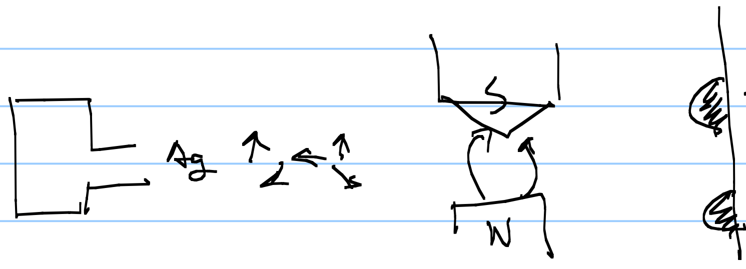
$$S_z |s\, m_s\rangle = \hbar m_s |s\, m_s\rangle$$



here $s \le m_s \le -s$

$\frac{1}{2} \le m_s \le -\frac{1}{2}$ $\quad m_s = \pm\frac{1}{2}$

This is exactly the space quantization S-G saw



$= +S_z = +\frac{1}{2}\hbar = |\uparrow\rangle$

$-S_z = -\frac{1}{2}\hbar = |\downarrow\rangle$

{ This <u>cures</u> all the spectroscopic problems!

Finally matrix representations of Spin!

Since $|s m_s\rangle$ = Finite, wavefunctions not so handy

build Complex Abstract Vector Space 2-D

say $S_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$

$S_z|\uparrow\rangle = +\frac{\hbar}{2}|\uparrow\rangle$

$S^2|\uparrow\rangle = \hbar^2\frac{1}{2}(\frac{1}{2}+1)|\uparrow\rangle$

$S^2|\downarrow\rangle = \hbar^2\frac{1}{2}(\frac{1}{2}+1)|\downarrow\rangle$

build matrix rep from abstract rep

$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Clearly $\langle\uparrow|\uparrow\rangle = 1$

$\langle\downarrow|\downarrow\rangle = 1$

$\langle\uparrow|\downarrow\rangle = 0$

$\langle\downarrow|\uparrow\rangle = 0$

Now lets see if can come up w/ some abstract spin $\hat{O}$'s!

we want $S_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$

or

$$\underbrace{S_z}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$$

Try $\underbrace{S_z} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

Then

$$\underbrace{\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}}_{S_z}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\underbrace{S_z}|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$$

Try $\underbrace{S_z}|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$

$$\begin{pmatrix}\frac{\hbar}{2} & 0\\0 & -\frac{\hbar}{2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-\frac{\hbar}{2}\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

yes!

Can go on to S_x and

$$S_z = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_z$$

$$S_y = \begin{pmatrix} 0 & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

$$S_z = \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x$$

where $\vec{\sigma} = [\sigma_x, \sigma_y, \sigma_z]$

= Pauli Spin Matrices!