

Chpt 7 of Sherrin.

IDEA:

$$\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

↓ energy eigenstates

$$\hat{H}\Psi = E\Psi$$

$$\Psi = N e^{-\frac{iEt}{\hbar}} \psi \quad \text{w/ Born's Normalization Condition}$$

$$\int_{-\infty}^{+\infty} \Psi^* \Psi dx = 1$$

ie square integrable

These requirements on Ψ , ψ , guarantee that the solutions ψ

form a complete Basis in Function Space w/ all props of vector space
ie Hilbert space.

So we have

Vector Space

Func Space

$$|\vec{A}|^2 = \vec{A} \cdot \vec{A} \iff \int_{-\infty}^{+\infty} \psi^* \psi dx \equiv (\psi | \psi)$$

w/ Q.M. calc's like

$$\langle \hat{O} \rangle = \int \psi^* \hat{O} \psi dx = \int \hat{O}^\dagger \psi^* \psi dx$$

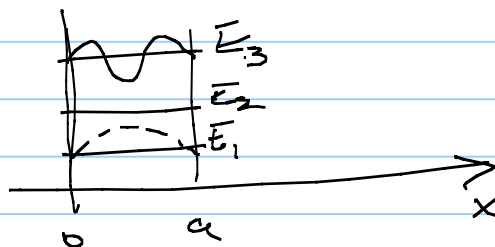
$$\hat{O}^\dagger = \text{adjoint} = (\hat{O}^T)^*$$

Great: Solns to Q.H. are wavefunctions that may be complex (but recall measurable $\psi^* \psi$ so complex wipos out there... but not always)

That works Great (Better than Great) for "continuous" measured quantities.

But in 1925, Goudsmit & Ullhenbeck (give PT article!) change things!

Up to now, we discovered Energy could be quantized



But (x) space was continuous ($0 \leq x \leq a$)

1925 G & U discover Space Quantization of SPIN!

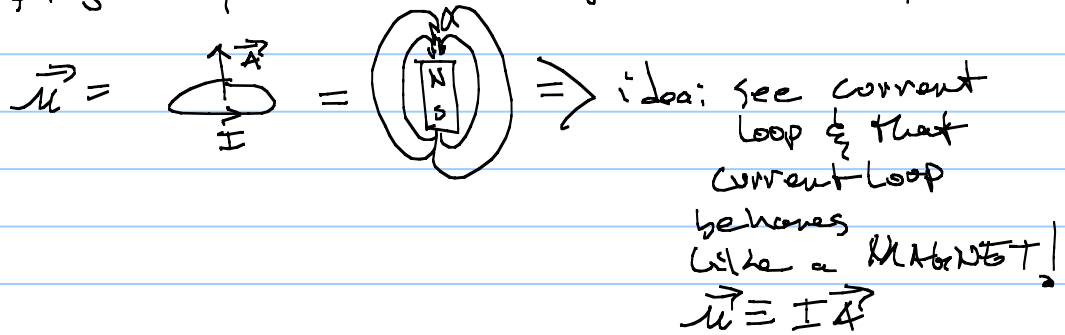
Again, entirely NON Classical.

Here is the story $\leftarrow \dots \rightarrow$

* This is also chpt 8 material!

1925: G & U space quantiz of spin.

They played w/ Torricelli's on magnetic dipoles, $\vec{\mu}$



so if you have an e^-

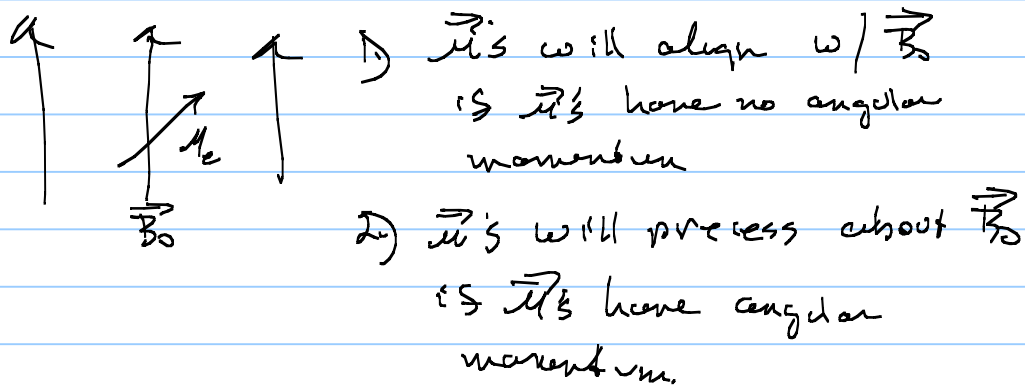
well, spinning e^- .



so

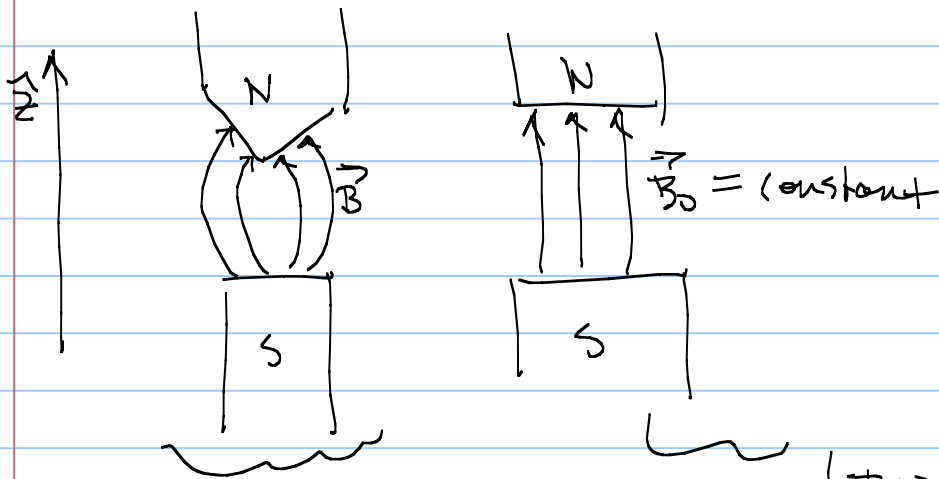
an e^- is tiny magnet =

If you put an e^- in a constant \vec{B}_0 field



So an e^- in a \vec{B}_0 will precess about \vec{B}_0 w/ precession frequency $\omega = \frac{g_e \mu_B}{2m_e} B_0$ = well known can tune into (= ESR)

.... well, that's all a different subject....
 I do something nuts



here you clearly
 don't have $B(z) = \text{constant}$

you have specially
 varying $\vec{B}(z)$

so $\frac{d\vec{B}(z)}{dz} \neq 0$

$$\frac{d\vec{B}(z)}{dz} = 0$$

$$= \text{constant}$$

$$\vec{B}(z)$$

so what will

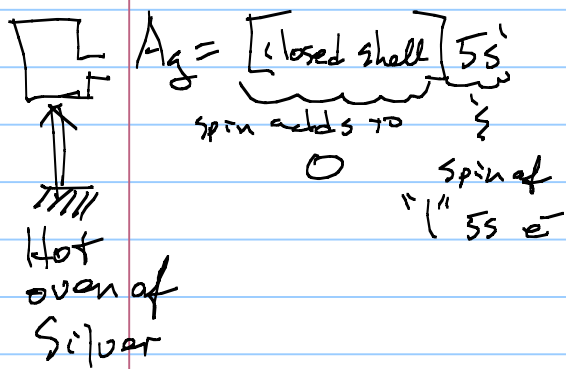
\vec{M}_S experience torque
 so precess (But don't
 translate)

while

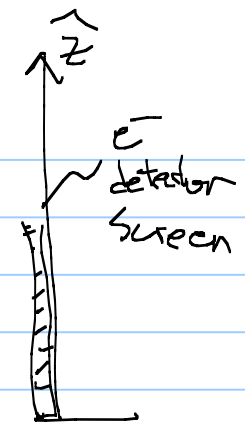
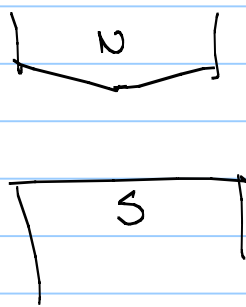
\vec{M}_S experience a Force & thus
 TRANSLATE ie Move!

So: G & U used this $\frac{d\vec{B}(z)}{dz} \neq 0$ trick
 called a Stern Gerlach experiment

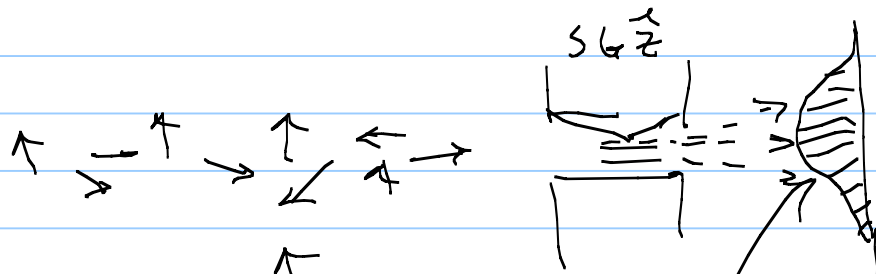
Strom Gerlach...



S.G. in \hat{z}



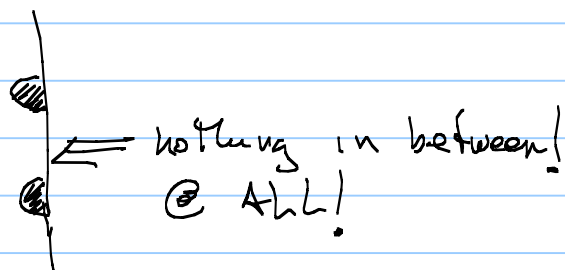
Here is classical idea of e⁻'s \Rightarrow \uparrow = \uparrow the



Nice \cong Flat distribution
 is \approx one 10^{23} randomly
 distribute Ag atoms

But They Found....

Instead



Classically expected \uparrow \downarrow all possibilities But

Q.M. result = \uparrow or \downarrow That's it!

AND... it didn't matter the direction of the S.G. device.

No matter \uparrow , e^- 's once the \vec{B}_0 was put on

could only be in 1 of 2 states up or down that's it.

This is called Space Quantization.
just as Energy is Quantized
 \uparrow exist only in steps

Spins can only exist in steps

Q.K. Back to Math

We just showed how it is no longer true that are Q.M. variables such as $x, \frac{d}{dt}$ are all continuous.

Indeed, we also have spin $\frac{1}{2}$ it is either \uparrow or \downarrow .

Now our mathematical formalism

$$\left. \begin{aligned} \hat{H}\psi &= E\psi \\ \int \psi^* \psi dx &= 1 \\ \langle \vec{0} \rangle &= \int \psi^* \vec{0} \psi dx \end{aligned} \right\} \text{Hilbert Space}$$

works great for functions, but what now about spin, \uparrow or \downarrow ?

Well key is Hilbert space we've worked in acted like $\infty \rightarrow$ vector space.

May be Spin = 2-D vector space?
is it complete? yes

So Idea is

In Finite Vector Space
Instead of infinite function space
 $\left. \begin{aligned} \psi &\Rightarrow \text{vectors} \\ \hat{O} \hat{S} &\Rightarrow \text{matrices} \end{aligned} \right\} \text{But these can all be made as matrices.}$

lets see....

$\Psi = \text{vectors}$
 $\hat{O} = \text{matrices}$ } matrices

that must act, just like our
Schrodinger Hilbert spaces
so will need $(\Psi|\Psi)$

$$\xi (\Psi|\hat{O}|\Psi) = (\hat{O}^T \Psi|\Psi)$$

OK. lets try this

$|\uparrow\rangle$
BRA

= state vector

$$\text{so } |\Psi\rangle = |\uparrow\rangle$$

psi = state vector

now instead of $\Psi(x) = \text{state wave function}$

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xi |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

OK... lets make this work.

$$\text{recall } \int_{-\infty}^{+\infty} \Psi^* \Psi dx = (\Psi|\Psi) = 1 \quad \xi |\vec{A}|^2 = \vec{A} \cdot \vec{A}$$

v.s.

so what to do w/ $|\uparrow\rangle$?

well want $|\Delta|^2 = \vec{A} \cdot \vec{A}$ or $\langle \psi | \psi \rangle = \int_{-\infty}^{+\infty} \psi^* \psi dx = 1$
v.s. Funct. Space

now here

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{column vector}$$

now about

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1 \times 1) + (0 \times 0) = 1$$

That worked --- so invent

$\langle \uparrow |$
ket

whose job is to match up
w/ a bra

$\langle \uparrow | \uparrow \rangle$ to form a "BRA-KET"

whose equiv is

vector \in Funct Space

dot prod

$$\langle \uparrow | \uparrow \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

That's it: here's the most general
rule

$|\uparrow\rangle = |kets = \text{column vectors}$

recall in Q.M. things often are complex so
 $|z|^2 = z^* z$

so to make a

BRA $\equiv \langle \uparrow |$

we needed to ① go from column vector to row vector

② include possibility that z 's may be complex.

so

column vector

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is column vector = $|\langle i; j \rangle = \begin{pmatrix} A_{11} \\ A_{12} \end{pmatrix}$

column *row*

then row vector of A_{ij} is
interchanges $i \leftrightarrow j$

$$\langle R_{ij} | = \langle \begin{matrix} i \\ j \end{matrix} | = \begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$$

column *row*

so

is

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ to make } |\downarrow\rangle \text{ you}$$

simply interchange the rows

$\frac{1}{2}$ columns! BUT - . . .

Don't forget need to ensure $|z|^2 = z^* z$
So must also complex conjugate.

So: if $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
then

$$\langle \uparrow | = \begin{pmatrix} 1 & 0 \end{pmatrix}^* = \begin{pmatrix} 1^* & 0^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

So that indeed

$$\langle \uparrow | \uparrow \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

Let give it a definition

$$\langle \uparrow | = \left[\begin{pmatrix} | \uparrow \rangle \end{pmatrix}^{\text{Transposed}} \right]^*$$

$$= | \uparrow \rangle^\dagger$$

adjoint!

We Now know exactly what transposed means!

Before in Function Space we were not too sure

For example

in Function space we saw ...
to make

$$\langle 0 \rangle = \int_{-\infty}^{+\infty} \psi^* \hat{0} \psi dx = \int_{-\infty}^{+\infty} [\hat{0}]^? \psi^* \psi dx$$

$= (\psi | \hat{0} | \psi) = (\hat{0}^? \psi | \psi)$ we had to make

$$[\hat{0}]^? \rightarrow \hat{0}^{T*} = \hat{0}^{\dagger}$$

↑
adjoint

recall ex: 5.5 pg 105 on how we had to do that w/ ψ 's = functions.

Now we have $|\psi\rangle$'s = matrices. Thus we will do

$$\langle 0 \rangle = \langle \psi | \hat{0} | \psi \rangle$$

$$= \begin{pmatrix} \psi_{11}^* & \psi_{12}^* \end{pmatrix} \begin{pmatrix} 0_{11} & 0_{12} \\ 0_{21} & 0_{22} \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix}$$

which we will want to = $\langle \hat{0} \psi | \psi \rangle$

$$= \begin{pmatrix} 0_{11}^* & 0_{21}^* \\ 0_{12}^* & 0_{22}^* \end{pmatrix} \begin{pmatrix} \psi_{11}^* & \psi_{12}^* \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix}$$

where $\hat{0}$ in bra space simply = $(\hat{0}^T)^* =$ same thing!

To conclude:

For Q.M. in finite-D space, the entire Formalism in Hilbert function space \leftrightarrow trans form one to one \leftrightarrow (DIRAC Matrix Formalism)

$\Psi = \sum c_n \psi_n$
WAVE Formalism

$$\langle \Psi | \Psi \rangle = \int \Psi^* \Psi dx = 1$$

\uparrow
 had to invent this

$\Psi = |\Psi\rangle = \text{column matrix}$
Matrix Formalism

$$\langle \Psi | \Psi \rangle; \langle \Psi | = (|\Psi\rangle)^T^*$$

\uparrow
 had to invent this

Complex Abstract Vector space possible yes expressed as matrices

$$\langle \Psi | \hat{O} | \Psi \rangle = \langle \hat{O} \Psi | \Psi \rangle = \langle 0 | \rangle$$

\uparrow
 needed to \hat{O}^T

$$\langle \Psi | \hat{O} | \Psi \rangle = \langle \hat{O} \Psi | \Psi \rangle$$

\uparrow
 \hat{O}^T

Note: $[A, B] \neq \text{vec zero}$

Note: $[A, B] = AB - BA$ also

not nec = zero so preserves commutation relations! matrices

Functions

\hat{O} 's change functions

\hat{O} 's = matrices rotate vectors (matrices)

ex: $\frac{d}{dx} x^2 = 2x$

Now they in Q.M. was Eigen Problem!

$$\hat{O} \psi = o \psi$$

\uparrow eigen function Not changed!
 \uparrow eigen value = multiplicative constant!

$$\hat{O} |\psi\rangle = o |\psi\rangle$$

\uparrow eigen vector NOT Related!
 \uparrow eigen value = how much stretched or shrunk

Now as eigen problem was so General
in Wave Formalism, clearly

$$\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

Then demanded Energy eigen State

$$\hat{H}\Psi = E\Psi \} = \text{Eigen Problem.}$$

Thus in Schrödinger Wave Formalism Q.M. was
about Solving Eigen Functions &
Eigen Values

In matrix formalism it is pretty much
the same

$$\hat{H}|\Psi\rangle = E|\Psi\rangle$$

But goal is to find

eigen vectors

↳ eigen values.

EXAMPLE: $\hat{H}|\psi\rangle = E|\psi\rangle$

$|\psi\rangle =$ column vectors

$\hat{H} =$ matrix

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix} = E \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix}$$

rewrite

$$\boxed{\hat{H}} |\psi\rangle = E |\psi\rangle$$

$$\left(\boxed{\hat{H}} - E \right) |\psi\rangle = 0$$

make
matrix

$$\boxed{E} = E \boxed{1}$$

identity
matrix = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\left(\boxed{\hat{H}} - \boxed{E} \right) |\psi\rangle = 0$$

instead of $\psi'' + \lambda\psi = 0$ ← Function problem.

OK

$$\underbrace{(H - E)}_{\text{for } |\psi\rangle \neq 0} |\psi\rangle = 0$$

$$\begin{pmatrix} H_{11} - E & H_{12} \\ H_{21} & H_{22} - E \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix} = 0$$

so need $\text{for } \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ←

$\underbrace{A}_{\text{matrix}} |\psi\rangle = 0$ when $\underbrace{A}_{\text{matrix}} = H - E$

But tricky

But here is the deal. (Toungsiend)

if $|A| \neq 0$ $|A| \equiv$ determinant of A

then $\underbrace{A}_{\text{matrix}}$ has an inverse

$$\underbrace{A^{-1}}$$

If $\underbrace{A}_{\text{matrix}}$ has an inverse then could always

$$\underbrace{A^{-1} A}_{\text{matrix}} |\psi\rangle = \underbrace{A^{-1}}_{\text{matrix}} (0)$$

$$\mathbb{1} |\psi\rangle = 0$$

$$|\psi\rangle = 0 = \text{(contradiction)}$$

So only way for

$$\underbrace{A}_{\text{matrix}} |\psi\rangle = 0 \quad \& \quad |\psi\rangle \neq 0$$

must be for $\underbrace{A}_{\text{matrix}}$ not to have an inverse

The only way \hat{A} can't have an inverse
is if

$$|\hat{A}| = 0$$

∴ To solve $\hat{H}|\psi\rangle = E|\psi\rangle$

↓

$$(\hat{H} - E)|\psi\rangle = 0$$

$$\hat{A}|\psi\rangle = 0$$

is if

$$\det |\hat{A}| = 0$$

or

$$\begin{vmatrix} H_{11} - E & H_{12} \\ H_{21} & H_{22} - E \end{vmatrix} = 0$$

That's how we will solve

Eigenvalue & Eigen vector
Matrix formalism of
QM

Quick example from Schenker:

$$A \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{Then } A|\psi\rangle = c|\psi\rangle$$

$$(A - c)|\psi\rangle = 0$$

$$(A - \mathbb{1}c)|\psi\rangle = 0$$

$$\begin{pmatrix} A_{11} - c & A_{12} \\ A_{21} & A_{22} - c \end{pmatrix} |\psi\rangle = 0$$

$$\begin{pmatrix} -c & 1 \\ 1 & -c \end{pmatrix} |\psi\rangle = 0$$

$$\text{For } |\psi\rangle \neq 0$$

Then

$$\det \begin{vmatrix} -c & 1 \\ 1 & -c \end{vmatrix} = 0$$

$$(-c)(-c) - (1)(1) = 0$$

$$c^2 = 1$$

$c = \pm 1$ } These are the eigenvalues!

$$A|\psi\rangle = \pm 1|\psi\rangle$$

now find eigen vectors!

Just substitute one eigen vector @ a time

$+1$

$$A|\psi\rangle = +1|\psi\rangle$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (+1) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\cancel{(0)}\psi_1 + (1)\psi_2 = \psi_1$$

$$(1)\psi_1 + \cancel{(0)}\psi_2 = \psi_2$$

$$\left. \begin{aligned} \psi_2 &= \psi_1 \\ \psi_1 &= \psi_2 \end{aligned} \right\}$$

will always end up this way... free to choose
↳ then can solve the other

Thanks it, let $\psi_1 = 1$
then $\psi_2 = 1$

$$\left\{ \begin{aligned} |\psi\rangle &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ +1 \end{aligned} \right.$$

? Don't worry about

in then and will need to

make

$$\langle\psi|\psi\rangle = 1$$

So our guess will not be arbitrary!

Set $\boxed{-1}$ $A|\psi_{-1}\rangle = (-1)|\psi_{-1}\rangle$ eigenvalue
for eigen
value (-1)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = -1 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\psi_2 = -\psi_1$$

$$\psi_1 = -\psi_2$$

say $\psi_1 = 1$
Then $\psi_2 = -1$

$$|\psi_{-1}\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Coob

$$|\psi_{+1}\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\psi_{-1}\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

now eventually will want these state functions to be normalized so

lets say $|\psi_{+1}\rangle = N \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Then $\langle \psi_{+1} | \psi_{+1} \rangle = 1$

$$(N^* \ N^*) \begin{pmatrix} N \\ N \end{pmatrix} = 1$$

$$N^* N + N^* N = 1$$

$$2N^2 = 1 \ ; \ N = \frac{1}{\sqrt{2}}$$

$$|\psi_{+1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\text{and } \langle \psi_+ | \psi_- \rangle = 1$$

$$(N^* - N^*) \begin{pmatrix} N \\ -N \end{pmatrix} = 1$$

$$N^*N + N^*N = 1$$

$$N = \frac{1}{\sqrt{2}}$$

$$|\psi_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

So Note ... see if $|\psi_+\rangle \perp |\psi_-\rangle$ or 1

well expect

$$\vec{1} \cdot \vec{0} = 0$$

so

$$\langle \psi_+ | \psi_- \rangle$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} - \frac{1}{2} = 0 \quad \text{yeah!}$$

GREAT!

$-E_{int} e^{ix}$

To Conclude:

Q.M.

$$\hat{H}\psi = i\hbar \frac{d\psi}{dt}$$

demand energy eigenfunctions or eigen vectors } eigenstate

This step separates time out which is continuous so will have function soln.

Hilbert, complete Function space for continuous variable

$$\hat{H}\psi = E\psi$$

$\psi'' + A\psi = 0$
solve d.s.s by Q

$$\Psi = e^{-\frac{E_{int}}{\hbar} x} \psi(x)$$

Finite vector space (complex, abstract (bra-kets?))

$(\hat{H} - E)|\psi\rangle = 0$
solve $\langle \psi | \hat{H} - E | \psi \rangle = 0$

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$\Psi = e^{-\frac{E_{int}}{\hbar} x} |\psi\rangle$$

$$\langle \psi | \psi \rangle = \int \psi^* \psi dx = \int \psi^* \psi dx$$

$$\langle \psi | \psi \rangle = \langle \psi | \hat{1} | \psi \rangle = \langle \hat{1} \psi | \psi \rangle$$

$\langle \psi | = (\psi)^* \dagger$

$$[\hat{H}, \hat{A}] = 0$$

ψ = simultaneous eigen funcs of $\hat{H} \ \& \ \hat{A}$

$$[\hat{H}, \hat{A}]$$

$|\psi\rangle$ = simultaneous eigen vectors of $\hat{H} \ \& \ \hat{A}$

Observable's

$$\hat{O}^\dagger = \hat{O}^T \dagger = \hat{O}$$

ie \hat{O} is Hermitian!

For Both formalisms!