The Function space of Schrö's ψ to CAVS of Dirac

Chapt 7 of Sherrer.

IDEA:

w/ $\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$

$\downarrow$ engy eigenstates

$\hat{H}\psi = E\psi$

$\Psi = N e^{-\frac{iE_n t}{\hbar}}\psi$ w/ Born's Normalization Condition

$$\int_{-\infty}^{+\infty}\Psi^*\Psi\,dx = 1$$

i.e. square integrable

Those requirements on psi, Ψ, guarantee that the solutions Ψ form a complete Basis in Function Space w/ all props of vector space

$\therefore$ Hilbert space.

So we have

Vector Space	Funct Space
$\lvert\vec{A}\rvert^2 = \vec{A}\cdot\vec{A}$ $\Longleftrightarrow$	$\int_{-\infty}^{+\infty}\psi^*\psi\,dx \equiv (\psi\vert\psi)$

w/ Q.M. calc's like

$$\langle\hat{O}\rangle = \int\psi^*\hat{O}\psi\,dx = \int\hat{O}^\dagger\psi^*\psi\,dx$$

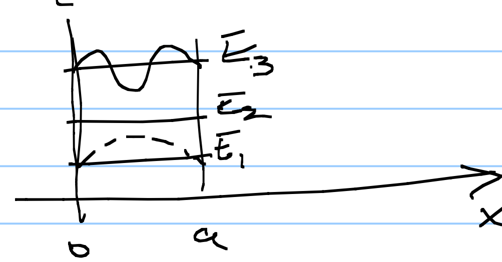
$\hat{O}^\dagger$ = adjoint $= (\hat{O}^T)^*$

Great: Solns to Q.M. are nice wavefunctions that may be complex
(but recall measurable $\propto \psi^*\psi$ so complex wipes out there... but not always)

That works Great (Better than Great) for "continuous" measured quantities.

But in 1925, Goudsmit & Uhlenbeck (give PT article!) change things!

Up to now, we discovered Energy could be quantized



But (x) space was continuous ($0 \le x \le a$)

1925 G & U discover Space Quantization of SPIN!

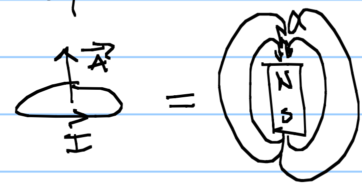
Again, entirely NON Classical.

Here is the story ---->

*This is also chapt 8 material!

1925: G & U space quantiz of spin.

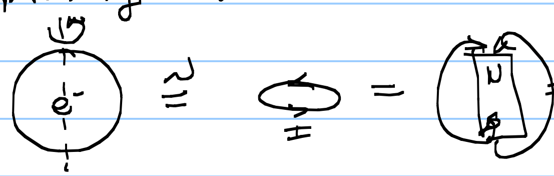
They played w/ forces on magnetic dipoles, $\vec{\mu}$

$\vec{\mu}$ =  ⟹ idea: see current loop & that current loop behaves like a MAGNET!

$\vec{\mu} \equiv I\vec{A}$

So if you have an e^-

well, spinning e^-?

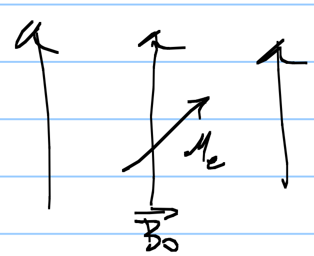


$= \vec{\mu}_e = \frac{g_e q_e}{2m_e}\vec{S} = \uparrow \vec{\mu}_e$

So

an e^- ≅ tiny magnet  =

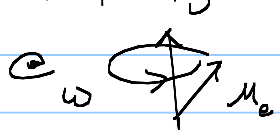
If you put an e^- in a constant $\vec{B}_0$ field



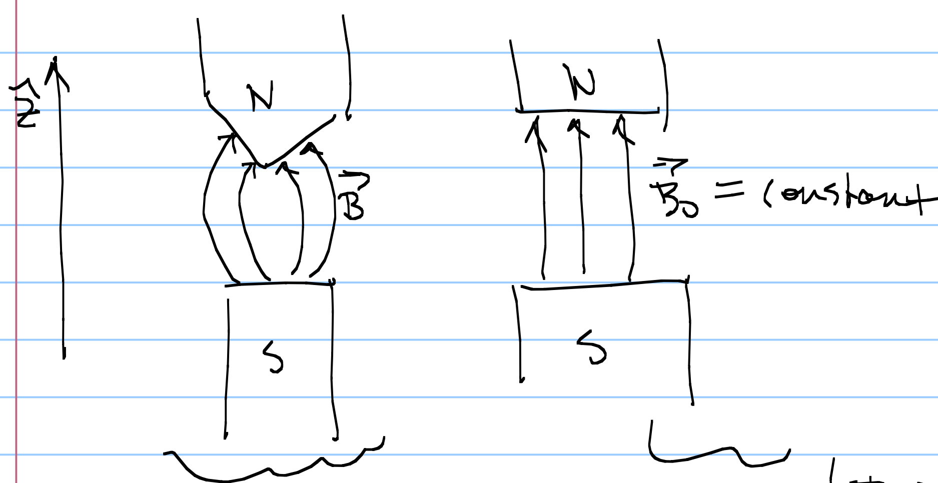
1) $\vec{\mu}$'s will align w/ $\vec{B}_0$ if $\vec{\mu}$'s have no angular momentum

2) $\vec{\mu}$'s will precess about $\vec{B}_0$ if $\vec{\mu}$'s have angular momentum.

So an e^- in a $\vec{B}_0$ will precess about $\vec{B}_0$ w/ precession frequency $\omega = \frac{g_e q_e B_0}{2m_e}$ = well known can tune into (= ESR)



.... well, that's all a different subject....
is do something nuts



here you clearly
don't have $B(z)$ = constant

you have spacially
varying $\vec{B}(z)$

so $\frac{d\vec{B}(z)}{dz} \neq 0$

$\frac{dB(z)}{dz} = 0$
= constant
$B(z)$

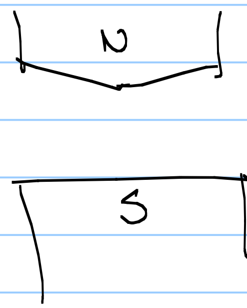
So what will $\vec{\mu}_s$ experience Torque
so precess (But do not
translate)

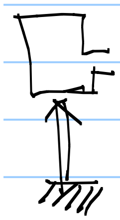
while $\vec{\mu}$'s experience a Force & thus
TRANSLATE .... ie Move!

So: G & U used this $\frac{dB(z)}{dz} \neq 0$ trick
called a Stern Gerlach experiment

Stern Gerlach...

S.G. in $\hat{z}$

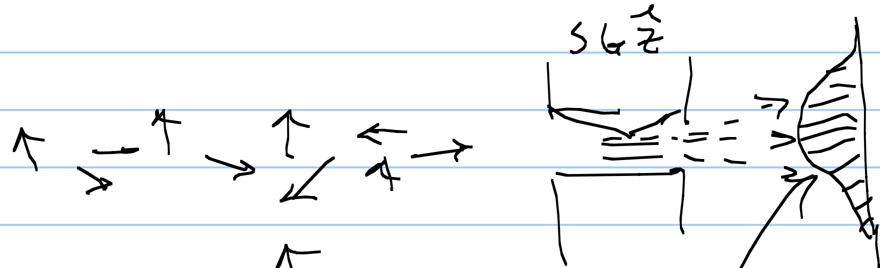




$Ag = \underbrace{[\text{closed shell}]}_{\text{spin adds to } 0}\ \underbrace{5s^1}_{\text{spin of "1" } 5s\ e^-}$

Hot oven of Silver

Here is classical idea of e^-'s $\cong$ $\boxed{\begin{matrix} N \\ S \end{matrix}}$ $= \uparrow \mu_e$



Nice $\cong$ Flat distribution is are 10^{23} randomly distribute Ag atoms

But They Found....

Instead

$\Longleftarrow$ nothing in between! @ ALL!

Classically expected $\nwarrow \uparrow \nearrow \rightarrow \swarrow \downarrow \searrow$ all possibilities But

Q.M. result = $\uparrow$ or $\downarrow$ Thats it!

AND.... it didn't matter the direction of the S.G. device.

No matter ↑, e^-'s once the $\vec{B_0}$ was put on

could only be in 1 of 2 states up or down thats it.

This is called Space Quantization.
just as Energy is Quantized & exhist only in steps

Spins Can only exist in steps

Q.M. Back to Math

We just showed how it is no longer true that are Q.M. variables such as

x, & t are all continuous.

Indeed, we also have spin & it is either $\uparrow$ or $\downarrow$.

Now our mathematical formalism

$$\left.\begin{array}{l} \hat{H}\psi = E\psi \\ \int \psi^* \psi \, dx = 1 \\ \text{\& } \langle \hat{O} \rangle = \int \psi^* \hat{O} \psi \, dx \end{array}\right\} \text{Hilbert Space}$$

works great for functions, but what now about spin, $\uparrow$ or $\downarrow$?

Well key is Hilbert space we've worked in acted like ∞-D vector space.

Maybe Spin = 2-D vector space?
is it complete? yes

So Idea is

In Finite Vector Space
Instead of in Infinite Function Space

$$\left.\begin{array}{l} \Psi \Rightarrow \text{vectors} \\ \text{\& } \hat{O}\text{'s} \Rightarrow \text{matrices} \end{array}\right\} \text{But these can all be made as matrices.}$$

Lets see....

$$\underbrace{\left.\begin{array}{l}\Psi = \text{vectors} \\ \hat{O} = \text{matrices}\end{array}\right\} \text{matrices}}$$

That must act just like our Schrodinger Hilbert spaces so will need $(\psi|\psi)$

$\&$ $(\psi|\hat{O}|\psi) = (\hat{O}^{T*}\psi|\psi)$

O.K. lets try this

$$\underbrace{\underbrace{|\uparrow\rangle}_{\text{BRA}}}_{}$$

= state vector

so $|\Psi\rangle = |\uparrow\rangle$

psi = state vector
now instead of $\psi(x)$ = state wave function

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\& \; |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

OK.... lets make this work.

recall $\int_{-\infty}^{+\infty} \psi^* \psi \, dx \equiv (\psi|\psi) = 1$ $\quad \& \quad |\vec{A}|^2 = \vec{A} \cdot \vec{A}$
V.S.

So what to do w/ $|\uparrow\rangle$?

well want $\underbrace{|A|^2 = \vec{A} \cdot \vec{A}}_{\text{V.S.}}$ or $\underbrace{\langle \psi | \psi \rangle = \int_{-\infty}^{+\infty} \psi^* \psi \, dx = 1}_{\text{Funct. Space}}$

now have

$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ = column vector

how about

$(1 \;\; 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1)(1) + (0)(0) = 1$

That worked --- so invent

$\underbrace{\langle \uparrow |}_{\text{ket}}$ whose job is to match up w/ a bra

$\langle \uparrow | \uparrow \rangle$ to form a "BRA-KET"

whose equiv is

Vector & Funct Space

dot prod

$\langle \uparrow | \uparrow \rangle = (1 \;\; 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Thats it: here's the most general rule

$|\uparrow\rangle$ = kets = column vectors

(recall in Q.M. things often are complex so
$|z|^2 = z^* z$)

so to make a
BRA $\equiv \langle\uparrow|$

we needed to ① go from column vector to row vector
② include possibility that #'s may be complex.

So

Column vector
$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is column vector $= |C_{ij}\rangle = \begin{pmatrix} A_{11} \\ A_{12} \end{pmatrix}$

column   row

Then row vector of A_{ij} is interchange i & j

$$\langle R_{ij}| = C_{ji} = \begin{pmatrix} A_{11} & A_{12} \end{pmatrix}$$

column   row

So if $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to make $|\downarrow\rangle$ you simply interchange the rows & columns! BUT - - -

don't forget need to ensure $|z|^2 = z^* z$
So must also complex conjugate.

So.. if $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
then

$$\langle\uparrow| = \begin{pmatrix} 1 & 0 \end{pmatrix}^* = \begin{pmatrix} 1^* & 0^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

So that indeed

$$\langle\uparrow|\uparrow\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

Let give it a definition

$$\langle\uparrow| = \left[\left(|\uparrow\rangle\right)^{\text{Transposed}}\right]^*$$

$$= |\uparrow\rangle^{\dagger}$$

adjoint!

We Now know exactly what transposed means!

Before in function space we were not too sure

For example

in Function space we saw ...
to make

$$\langle O \rangle = \int_{-\infty}^{+\infty} \psi^* \hat{O} \psi \, dx = \int_{-\infty}^{+\infty} [\hat{O}]^{?} \psi^* \psi \, dx$$

$= (\psi|\hat{O}|\psi) = (\hat{O}^{+}\psi|\psi)$ we had to make

$$[\hat{O}]^{?} \rightarrow \hat{O}^{T*} = \hat{O}^{+}$$

adjoint

& recall ex: 5.5 pg 105 on how we had to do that w/ ψ's = functions.

Now we have $|\psi\rangle$'s = matrices. Thus we will do

$$\langle O \rangle = \langle \psi | \hat{O} | \psi \rangle$$

$$= \begin{pmatrix} \psi_{11}^* & \psi_{12}^* \end{pmatrix} \begin{pmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix}$$

which we will want to $= \langle \hat{O} \psi | \psi \rangle$

$$= \begin{pmatrix} O_{11}^* & O_{21}^* \\ O_{12}^* & O_{22}^* \end{pmatrix} \begin{pmatrix} \psi_{11}^* & \psi_{12}^* \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix}$$

where $\hat{O}$ in bra space simply $= (\hat{O}^{T})^{*} =$ same thing!

To conclude:

For Q.M. in Finite-D space, the entire Formalism (in Hilbert function space) transforms one to one (DIRAC Matrix Formalism)

ψ = funct — $\psi = |\psi\rangle$ = column matrice

WAVE Formalism — **Matrix formalism.**

$(\psi|\psi) \equiv \int \psi^* \psi \, dx = 1$ — had to invent this

$\langle\psi|\psi\rangle$; $\langle\psi| = (|\psi\rangle)^{T*}$ — had to invent this

Complex (possible) Abstract (ops) Vector space (expressed as matrices).

$(\psi|\hat{O}|\psi) = (\hat{O}\psi|\psi) = \langle O\rangle$ — needed to $\hat{O}^{T*}$

$\langle\psi|\hat{O}|\psi\rangle = \langle\hat{O}\psi|\psi\rangle$ — $\hat{O}^{T*}$

Note: $[\hat{A},\hat{B}] \neq$ nec zero — Functions

note $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ also not nec. = zero so preserves commutation relations! — matrices

$\hat{O}$'s change functions

ex: $\frac{d}{dx}x^2 = 2x$

$\hat{O}$'s = matrices rotate vectors (matrices)

How key in Q.M. was Eigen Problem!

$\hat{O}\psi = o\,\psi$ — eigen function Not changed! — eigen value = multiplicative constant!

$\hat{O}|\psi\rangle = o\,|\psi\rangle$ — eigen vector NOT Rotated! — eigenvalue = how much stretched or shrunk

Now as eigen problem was so General
in Wave formalism, clearly

$$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

Then demanded Energy eigen funct

$$\left.\hat{H}\psi = E\psi\right\} = \text{Eigen Problem.}$$

Thus in Schrö wave formalism Q.M. was
about solving Eigen Functions &
Eigen Values

In matrix formalism it is pretty much
the same

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

But goal is to find

eigen vectors

& eigen values.

EXAMPLE: $\hat{H}|\psi\rangle = E|\psi\rangle$

$|\psi\rangle$ = Column vectors

$\hat{H}$ = matrix

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix} = E \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix}$$

rewrite

$$\underline{H}\,|\psi\rangle = E\,|\psi\rangle$$

$$(\underline{H} - E)\,|\psi\rangle = 0$$

make matrix $\underline{E} = E\,\underline{\mathbb{1}}$

identity matrix $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$(\underline{H} - \underline{E})\,|\psi\rangle = 0$$

instead of $\psi'' + A\psi = 0$ $\Leftarrow$ Function problem.

OK

$$\left(\underset{\sim}{H}_s - \underset{\sim}{E}_s\right)|\psi\rangle = 0$$

$$\Uparrow$$

for $|\psi\rangle \neq 0$

$$\begin{pmatrix} H_{11}-E & H_{12} \\ H_{21} & H_{22}-E \end{pmatrix}\begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix} = 0$$

for $\begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

so need

$\underbrace{\underset{\sim}{A}_s |\psi\rangle = 0}$ when $\underset{\sim}{A}_s = \overset{\underset{\sim}{H}_s - \underset{\sim}{E}_s}{\text{matrix}}$

Bit tricky

But here is the deal. (Townsend)

if $|\underset{\sim}{A}_s| \neq 0$ $\quad |\underset{\sim}{A}_s| \equiv$ determinant of A

then $\underset{\sim}{A}_s$ has an inverse

$\underset{\sim}{A}_s^{-1}$

If $\underset{\sim}{A}_s$ has an inverse then could always

$$\underbrace{\underset{\sim}{A}_s^{-1}\underset{\sim}{A}_s}|\psi\rangle = \underset{\sim}{A}_s^{-1}(0)$$

$$\mathbb{1}|\psi\rangle = 0$$

$$|\psi\rangle = 0 = \text{contradiction}$$

So only way for $\underset{\sim}{A}_s|\psi\rangle = 0$ & $|\psi\rangle \neq 0$

must be for $\underset{\sim}{A}_s$ not to have an inverse

The only way $\underset{\sim}{A}$ can't have an inverse is if

$$|\underset{\sim}{A}| = 0$$

$\therefore$ To solve $\hat{H}|\psi\rangle = E|\psi\rangle$

$$\Downarrow$$

$$(\underset{\sim}{H} - \underset{\sim}{E})|\psi\rangle = 0$$

$$\underset{\sim}{A}|\psi\rangle = 0$$

is if

$$\det|\underset{\sim}{A}| = 0$$

or

$$\begin{vmatrix} H_{11} - E & H_{12} \\ H_{21} & H_{22} - E \end{vmatrix} = 0$$

Thats How we will solve

Eigenvalue & Eigenvector

Matrix formalism of

Q.M

Quick example from Scherrer:

$$\underset{\sim}{A} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $\underset{\sim}{A}|\psi\rangle = c|\psi\rangle$

$$(\underset{\sim}{A} - c)|\psi\rangle = 0$$

$$(\underset{\sim}{A} - \underset{\sim}{\mathbb{1}}c)|\psi\rangle = 0$$

$$\begin{pmatrix} A_{11} - c & A_{12} \\ A_{21} & A_{22} - c \end{pmatrix} |\psi\rangle = 0$$

$$\begin{pmatrix} -c & 1 \\ 1 & -c \end{pmatrix} \left(|\psi\rangle \right) = 0$$

for $|\psi\rangle \neq 0$

Then

$$\det \begin{vmatrix} -c & 1 \\ 1 & -c \end{vmatrix} = 0$$

$$(-c)(-c) - (1)(1) = 0$$

$$c^2 = 1$$

$$c = \pm 1$$

} These are the eigenvalues!

$\underset{\sim}{A_{\cdot}} |\psi\rangle = \pm 1 |\psi\rangle$

now find eigen vectors!

Just substitute one eigen vector @ a time

$\boxed{+1}$

$\underset{\sim}{A_{\cdot}} |\psi_{+1}\rangle = +1 |\psi_{+1}\rangle$ — use these to keep from getting confused

$|\psi_{+1}\rangle$ = eigenvector for eigenvalue (+1)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (+1)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\cancel{(0)\psi_1} + (1)\psi_2 = \psi_1$$

$$(1)\psi_1 + \cancel{(0)\psi_2} = \psi_2$$

$$\left.\begin{aligned} \psi_2 &= \psi_1 \\ \psi_1 &= \psi_2 \end{aligned}\right\}$$ will always end up this way... free to choose ↳ then can solve the other

That's it, let $\psi_1 = 1$

then $\psi_2 = 1$

$$\therefore \quad |\psi_{+1}\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

? Don't worry about in then and will need to make $\langle\psi|\psi\rangle = 1$

So our guess will not be arbitrary!

for $\boxed{-1}$ $\hat{A}_1 |\psi_{-1}\rangle = (-1)|\psi_{-1}\rangle$ — eigenvector for eigenvalue (-1)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = -1 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\psi_2 = -\psi_1$$
$$\psi_1 = -\psi_2$$

say $\psi_1 = 1$
Then $\psi_2 = -1$

$$|\psi_{-1}\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Cool

$$|\psi_{+1}\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$|\psi_{-1}\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

now eventually will want these state functions to be normalized so

lets say $|\psi_{+1}\rangle = N\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Then $\langle\psi_{+1}|\psi_{+1}\rangle = 1$

$$(N^* \;\; N^*)\begin{pmatrix} N \\ N \end{pmatrix} = 1$$

$$N^*N + N^*N = 1$$

$$2N^2 = 1 \;;\; N = \frac{1}{\sqrt{2}}$$

$$|\psi_{+1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

cont'd $\langle \psi_{-1} | \psi_{-1} \rangle = 1$

$$\begin{pmatrix} N^* & -N^* \end{pmatrix} \begin{pmatrix} N \\ -N \end{pmatrix} = 1$$

$$N^*N + N^*N = 1$$

$$N = \frac{1}{\sqrt{2}}$$

$$|\psi_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

(So Nots .... see if $|\psi_+\rangle$ & $|\psi_-\rangle$ are $\perp$

well expect

$$\vec{u} \cdot \vec{v} = 0$$

so

$$\langle \psi_+ | \psi_- \rangle$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}}^* & \frac{1}{\sqrt{2}}^* \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} - \frac{1}{2} = 0$$ yeah!

GREAT!

To Conclude:

Q.M.

$$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$e^{-\frac{E_n t}{\hbar}}$ This step separates time out which is continuous so will have function soln.

demand energy eigenFunctions or eigen Vectors } eigenstate

Hilbert, complete Function space for continuous variable

$\hat{H}\psi = E\psi$: $\psi'' + A\psi = 0$ solve diff Eq

$$\Psi = e^{-\frac{E_n t}{\hbar}}\psi(x)$$

Finite vector space (complex, abstract (bra-kets?)) vector space

$(\underset{\sim}{H} - E)|\psi\rangle = 0$ solve $\det|\underset{\sim}{H} - E| = 0$

$$\underset{\sim}{H}|\psi\rangle = E|\psi\rangle$$

$$\Psi = e^{-\frac{E_n t}{\hbar}}|\psi\rangle$$

$$\langle O\rangle = \int \psi^* \hat{O}\psi\, dx = \int \hat{O}^\dagger\psi^*\psi\, dx$$

$$\langle O\rangle = \langle\psi|\hat{O}|\psi\rangle = \langle\hat{O}^\dagger\psi|\psi\rangle$$

$$\langle\psi| = (|\psi\rangle)^{T*}$$

$$[\hat{H}, \hat{A}] = 0$$

ψ = simultaneous eigen funcs of $\hat{H}$ & $\hat{A}$

$$[\underset{\sim}{H}, \underset{\sim}{A}]$$

$|\psi\rangle$ = simultaneous eigen vectors of $\underset{\sim}{H}$ & $\underset{\sim}{A}$

Observerable's

$\hat{O}^\dagger = \hat{O}^{T*} = \hat{O}$ For Both the formalisms!

ie $\hat{O}$ is Hermetian!