E.F. Deveney / BSC Physics: PH402 Quantum I

Note Title 4/9/2006

Linear Algebra in Q.M. (part 2) Scherrer, C5

Linear $\hat{O}$, Commutation Relations & Uncertainty Princ.

1

We've already done Vector Spaces vs Function Spaces

VS vs FS

2

and Linear $\hat{O}$'s.

Here, we will expand a bit on both now that we've developed some of our formalism for Q.M. In doing so, we will build our ideas and mathematics more soundly.

Linear $\hat{O}$'s:

we already know

1) properties: $\hat{O}\left(c f(x)\right) = c\,\hat{O} f(x)$

$$\hat{O}\left[f(x) + g(x)\right] = \hat{O} f(x) + \hat{O} g(x)$$

2) In Q.M. it was noted that all observables (measurable) quantities can be built from Linear $\hat{O}$'s where

→ if $\hat{O}$ is eig. funct of Ψ

Then $\langle O \rangle \pm \cancel{\Delta O}$ = definite

→ if $\hat{O} \neq$ eig. funct of Ψ

then $\langle O \rangle \pm \Delta O$ not definite

but still measurable.

new on Linear $\hat{O}$'s:

properties: Let $\hat{R} = (\hat{P} \pm \hat{Q})$

1.) Then
$$\hat{R}\, f(x) = (\hat{P} \pm \hat{Q})\, f(x) = \hat{P} f(x) \pm \hat{Q} f(x)$$

2.) $\hat{R} = \hat{P}\hat{Q}$

then $\hat{R}\, f(x) = \hat{P}\left(\hat{Q}\, f(x)\right)$

Thats new stuff.
But Raises a ? about $\hat{O}$'s: Does order matter?

start w/ ordinary #'s a & b

$$ab\,(f(x)) \overset{?}{=} ba\,(f(x))$$ yes! of course

$(ab - ba)\, f(x) = 0$ = equiv statement

or

$$[a, b] \equiv \text{commutation of } a \;\&\; b \equiv ab - ba.$$

For ordinary #'s, a & b "Commute"
their commutation
relation is Zero.

ORDER does not
Matter.

Is this true of $\hat{O}$'s?

do $\hat{O}$'s commute?

lets try $\hat{H} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$

$\hat{p} = -i\hbar\frac{\partial}{\partial x}$

ask

$[\hat{H},\hat{p}] \stackrel{?}{=}$

$[\hat{H},\hat{p}] = (\hat{H}\hat{p} - \hat{p}\hat{H}) \stackrel{?}{=}$

to see, need to toss in a general test function! $f(x)$ why not call it $\psi(x)$

$(\hat{H}\hat{p} - \hat{p}\hat{H})\psi(x) =$

$= \left\{\hat{H}\left(-i\hbar\frac{\partial\psi}{\partial x}\right) - \hat{p}\left(\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi\right)\right\}$

$= \cancel{\frac{i\hbar^3}{2m}\frac{\partial^3\psi}{\partial x^3}} - \cancel{i\hbar V\frac{\partial\psi}{\partial x}} - \cancel{\frac{i\hbar^3}{2m}\frac{\partial^3\psi}{\partial x^3}} + i\hbar\psi\frac{\partial V}{\partial x} + \cancel{i\hbar V\frac{\partial\psi}{\partial x}}$

$= i\hbar\psi\frac{\partial V}{\partial x}$

So

$[\hat{H},\hat{p}]\cancel{\psi} = \left(i\hbar\frac{\partial V}{\partial x}\right)\cancel{\psi}$

note test funct gets wiped out

But really needed it to do this commutation (otherwise would have prob messed up)

$[\hat{H},\hat{p}] = i\hbar\frac{\partial V}{\partial x}$

$\hat{H}$ & $\hat{p}$ do not commute

Wow! ordinary #'s $\rightarrow$ order $\neq$ matter

$\hat{O}$'s $\longrightarrow$ order matters!

Here is the deal....

say $\hat{A}$ & $\hat{B}$ Both share the same eigen function ψ. (ψ contains definite info on a & b)

That means $\hat{A}\psi = a\psi$ a & b = definite!

$\hat{B}\psi = b\psi$

$\therefore [\hat{A},\hat{B}] \stackrel{?}{=}$ apply test ψ

$$[\hat{A}\hat{B}-\hat{B}\hat{A}]\psi = \hat{A}(\hat{B}\psi) - \hat{B}(\hat{A}\psi)$$
$$= \hat{A}(b\psi) - \hat{B}(a\psi)$$
$$= b\hat{A}\psi - a\hat{B}\psi$$
$$= ba\psi - ab\psi$$
$$= (\underbrace{ba - ab}_{\#'s})\psi = 0$$

So: if $\hat{O}$'s do commute, $[\hat{A},\hat{B}]=0$ It MEANS that they share the same eigenstates: Both observables are know Definitely for the eigen function.

On the Contrary: if $[\hat{A},\hat{B}] \neq 0$. It MEANS That the do not commute with each other and that they do not share the same eigenstates!

if you have: $\hat{A}\psi = a\psi$, $a \pm \underbrace{\Delta a}_{\to 0}$

Then $\hat{B}\psi = \underbrace{\psi'}_{\text{changes } \psi!}$ But $\langle b \rangle \pm \Delta b$ is still valid.

So you can kind of see how this happens

is

$$\hat{A}\psi = a\psi$$

but $\hat{B}\psi = \tilde{\psi}$ ← $\hat{B}$ not being an $\hat{O}$ of eigen function ψ changes ψ

$\hat{B}\psi \rightarrow \psi'$

That's what messes up the commutation!

then

$$[\hat{A}\hat{B} - \hat{B}\hat{A}]\psi$$

$$\hat{A}(\hat{B}\psi) - \hat{B}(\hat{A}\psi)$$

$$= \hat{A}(\psi') - \hat{B}(a\psi)$$

$$= \hat{A}(\psi' - a\hat{B}(\psi)$$

$$= \hat{A}(\psi') - ab\,\psi \neq 0 \text{ necessarily!}$$

Now, while not obvious Commutation Relations have deeper physical significance!

The Uncertainty Principle! = ROOT of problem.

Uncertainty Principle & Commutation Relation

$\underbrace{\text{Uncertainty Principle}}$

$\Downarrow$

True for all waves

& clearly Q.M. says $\psi_e = \underbrace{\text{wave}} + \text{particle}$

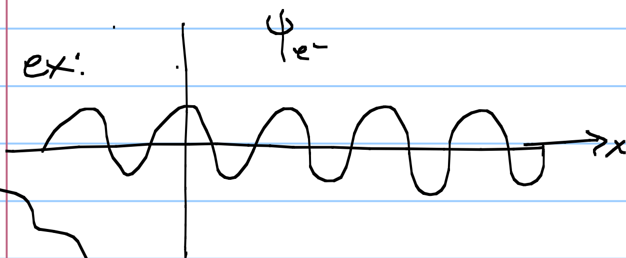
So will have this uncertainty.

$\psi_e \sim e^{i(kx-\omega t)}$

$k = 2\pi/\lambda$, $\omega = \frac{2\pi}{T}$

$p = \hbar k$, $E_T = \hbar\omega$

ex:



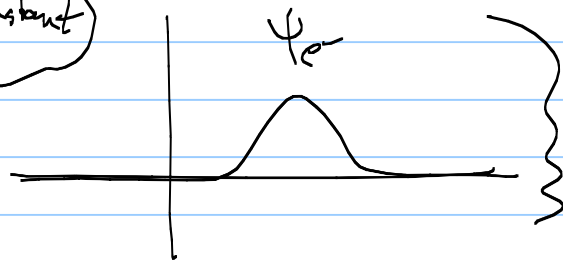
hold t = constant

clearly λ is well known

$\lambda \pm \underbrace{\Delta\lambda}_{\text{small}}$

But x of e^- ?

$x \pm \underbrace{\Delta x}_{\text{Big}}$



here $x \pm \underbrace{\Delta x}_{\text{small}}$

But $\lambda \pm \underbrace{\Delta\lambda}_{\text{Big!}}$

For all waves (see Harris) you can show

These two properties, position & wavelength play Complimentary Role

$$\Delta x \, \Delta k \geq \frac{1}{2} \qquad \text{where} \qquad k = \frac{2\pi}{\lambda} \; \& \; \underbrace{\frac{\Delta k}{k} = \frac{\Delta\lambda}{\lambda}}_{\text{by prop of errors}}$$

in Q.M. $|p| = \hbar k$

so $\Delta p = \hbar \Delta k$

so

$$\hbar\left(\Delta x \Delta k \geq \frac{1}{2}\right)$$

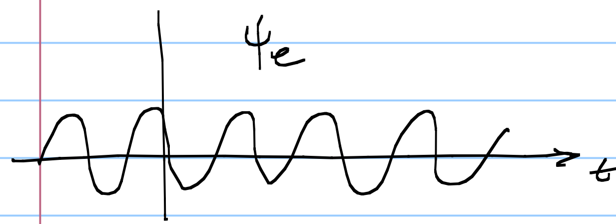
or $\Delta x (\hbar \Delta k) \geq \frac{\hbar}{2}$

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

with each other (now position & momentum). The more you know about one, the less you know about the other!

Complimentary pairs like these show up all the time in Q.M. Because of, ultimately, wave nature.

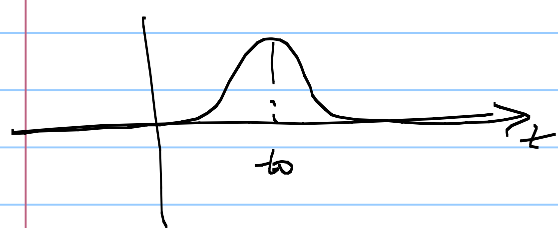
if you look @ ψ_e & hold x constant...



$f \pm \underbrace{\Delta f}_{\text{well known}}$

$t \pm \underbrace{\Delta t}_{\text{not}}$

or



$f \pm \underbrace{\Delta f}_{?}$

$t \pm \underbrace{\Delta t}_{\text{well known}}$

for all waves

$\Delta t \, \Delta \omega \geq \frac{1}{2}$ where $\omega = 2\pi f$

& $\Delta \omega = \Delta f$

since $E = \hbar \omega$

$\Delta E = \hbar \Delta \omega$

$\hbar (\Delta t \, \Delta \omega \geq \frac{1}{2})$

$\Delta E \, \Delta t \geq \hbar/2$

So what the uncertainty principle is telling us IS what in fact is coded in the commutation relations! Thats why it is so important!

so

$\Delta x \Delta p \geq \hbar/2$ can't measure x & p BOTH exactly --- hey sounds like x & p can't share the same eigen function which means their commutation should not be $= 0$

try! $[\hat{p}, \hat{x}] \overset{?}{=}$ try test funct ψ

$[\hat{p}\hat{x} - \hat{x}\hat{p}]\psi$ $\qquad \hat{p} = -i\hbar \frac{\partial}{\partial x}$

$p(\hat{x}\psi) - \hat{x}(\hat{p}\psi)$ $\qquad \hat{x} = x$

$-i\hbar\frac{\partial}{\partial x}(x\psi) - x\left(-i\hbar\frac{\partial}{\partial x}\psi\right)$

$-i\hbar\psi - \cancel{x\left(i\hbar\frac{\partial\psi}{\partial x}\right)} + \cancel{x\left(i\hbar\frac{\partial\psi}{\partial x}\right)}$

$[\hat{p}, \hat{x}]\cancel{\psi} = -i\hbar\cancel{\psi}$

$[\hat{p}, \hat{x}] = -i\hbar$ } They, $\hat{p}$ & $\hat{x}$ don't commute!

Now note

$$[\hat{H},\hat{p}] = i\hbar \frac{\partial V}{\partial x}$$

Then $[\hat{p},\hat{x}] = -i\hbar$

Well, $\hat{H}$ & $\hat{p}$ might commute if $\frac{\partial V}{\partial x} = 0$ ie constant potential

→ So if you have constant potential, you will be able to find a ψ that is eigenstate of both so that

$$\hat{H}\psi = E_T\psi$$

$$\& \ \hat{p}\psi = p\psi$$

BUT; $[\hat{p},\hat{x}] = -i\hbar$ NEVER can $= 0$

This is TRUE uncertainty principle underlies this non-commutation!

You will never get a ψ for both simultaneously

& ∴ will never be able to measure both definitively!

Finally! on commutations

we have been solving...

$$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$V = V(r)$ only (ie sep of variables)

Then demand $i\hbar\frac{\partial\Psi}{\partial t} = E_n\Psi$ ie Energy Eigen functions

then

$$\hat{H}\psi = E_T\psi$$

$$\& \ \Psi = e^{-\frac{iE_T t}{\hbar}}\psi$$

CLEARLY if we are solving Schrö time independ equation we are finding ψ & Ψ that are Energy eigen functions

Thus we are interested in knowing the $\hat{O}$'s that commute with total energy BECAUSE, they too will have the same eigen function (share it simultaneously) and can therefore be known Definitely!

ex: $[\hat{H},\hat{A}]=0$
$\& \ [\hat{H},\hat{B}]=0$
$[\hat{H},\hat{C}]=0$

$\Rightarrow$'s

$$\hat{H}\Psi = E_T\Psi$$
$$\hat{A}\Psi = a\Psi$$
$$\hat{B}\Psi = b\Psi$$
$$\hat{C}\Psi = c\Psi$$

these definites of energy eigenstates are called Good QUANTUM #s!

if we solve $\hat{H}\psi = E_T\psi$, for ψ,
then we
know E_T, a, b & c definitely!

Note also that this $\Rightarrow$'s

$$\hat{A}\psi = a\psi$$

this ψ is eigen funct of
$\hat{A}, \hat{B}, \hat{C}$ & $\hat{H}$ too

so can
get a, b, c & E

Because $[\hat{A},\hat{B}]=0$

$[\hat{A},\hat{C}]=0$

$[\hat{B},\hat{C}]=0$

follows from $\left[\hat{H}, \begin{matrix}\hat{A}\\ \hat{B}\\ \hat{C}\end{matrix}\right]=0$

Thus if know all commuting $\hat{O}$'s w/ $\hat{H}$,
Solve the easiest Schrö for ψ
you can find (ie easiest of commuting $\hat{O}$'s)