Scattering States Solns to Time Indep Schr"o: 

1) Free particle scattering states

- $E_T$

- $V(x)$

- Note $E_T > V$

- Always = free

- Is scattering? 

2) Scattering states

- $E_T$

- $V(x)$

3) Then Tunneling

- $V(x)$

- Alpha-decay

- Tunneling microscope

- Tunnel divide
I. Free Particle Scattering States in Detail!

Prototype for many other solutions

\[ \Psi(x) = \begin{cases} A e^{i \sqrt{2m(E_t-V)} x} + B e^{-i \sqrt{2m(E_t-V)} x} & \text{Region I: } V = 0 \vspace{0.2cm} \\ A_2 e^{i \left( \sqrt{2m(E_t-V_0)} x \right)} + B_2 e^{-i \left( \sqrt{2m(E_t-V_0)} x \right)} & \text{Region II: } V = V_0 \end{cases} \]

Energy (Scales) \[ E_{\text{tot}} \]

Region I

Region II

\[ x = 0 \]

It is very convenient to set up problems like this!

\[ E_t > V \text{ always (high energy states)} \]

\[ \Psi_0 = E_t \Psi \]

\[ \Psi'' + \frac{2m}{k^2} (E_t - V) \Psi = 0 \]

Now \[ E_t > V \text{ Always} \]

So

\[ \Psi(x) = C_1 e^{i \sqrt{2m(E_t-V)} x} + C_2 e^{-i \sqrt{2m(E_t-V)} x} \]

But 2 Regions SO

use same soln in both regions

\[ A_1 = \frac{2m}{k^2} E_t \]

\[ V = 0 \]

\[ A_2 = \frac{2m}{k^2} (E_t - V_0) \]

\[ V = V_0 \]
Now to make things a bit more clear,
recognize

$$\Gamma = \sqrt{\frac{2mE}{\hbar}} \quad \text{or} \quad \sqrt{\frac{2m(E-U)}{\hbar}} = \sqrt{\frac{\hbar^2 \Delta}{2m}}$$

$$E = E_f + E_p$$

$$E_f - V = \frac{p^2}{2m} \quad \Rightarrow \quad \frac{p^2}{2m} = \frac{\hbar^2 \Delta}{2m}$$

$$p = \hbar \Delta$$

or $$\Gamma = \text{WAVE #1} = k = \frac{2\pi}{\lambda}$$

$$\frac{\hbar}{m} \Delta = \frac{\hbar}{m} \times \frac{\hbar}{m} \times \frac{\hbar}{m} = \frac{1}{m}$$

$$\frac{\hbar}{m} \Delta$$

So $$\frac{E_f}{E}$$

I

$$V = 0$$

$$\text{x<0}$$

$$\psi(x) = A_1 e^{-k_1 x} + B_1 e^{-k_2 x}$$

II

$$0 \leq x$$

$$\psi(x) = A_2 e^{i k_1 x} + B_2 e^{-i k_2 x}$$

Now all we have to do is make

4 well behaved for all x such that $$\int_{-\infty}^{\infty} \psi^* \psi dx = \text{finite}$$

so well we know so ln's

For $$x = 0$$ is nice

So only place we have to work hard is @ x = 0

Need $$\psi_1(0) = \psi_2(0)$$

$$\frac{d\psi}{dx} \bigg|_{x=0} = \frac{d\psi}{dx} \bigg|_{x=0}$$
Before jumping in... try a physical argument to make life easier.

\[ V_{\text{bar}} \]

\[ e \rightarrow V_0 \]

Big point is that build e-gun & Free's →

Well in region I

\[ \psi(x) = A_1 e^{-i k_1 x} + B_1 e^{i k_1 x} \]

Recall \( \Phi(x) = e^{-i \frac{e}{m} \frac{x^2}{2}} \phi(x) \); Recall \( E_{\text{tor}} = \frac{e V_0}{m} \)

\[ \Phi(x) \]

\[ = A_1 e^{-i(k_1 x - wt)} + B_1 e^{i(k_1 x - wt)} \]

\[ = A_1 e^{-i(k_1 x - wt)} + B_1 e^{i(k_1 x + wt)} \]

\[ \rightarrow \]

\[ = \text{plane wave} + \text{plane wave} \]

\[ \rightarrow \]

\[ \Phi(x) = A_2 e^{i(k_2 x - wt)} + B_2 e^{-i(k_2 x - wt)} \]

\[ \rightarrow \]

\[ \text{note: } w = \text{same in I} \neq \text{II} \rightarrow \]

\[ \text{of course } E_t = k_0 \text{ doesn't change} \]

\[ \text{But } p = k \text{ does... so } k, k_1 \text{ kin (i.e. } k_1, k_2) \]
1) in I, might start up \[ \rightarrow \]
   But could have reflection \[ \leftarrow \]

2) in II, might get transmitted \[ \rightarrow \]
   but could not have particles \[ \leftarrow \]
   \[ \text{ie } B_2 e^{-ik_2 x - wt} \]
   \[ = \text{ plane wave} \leftarrow \text{ starting } @ +\infty \]

So .... Make \[ B_2 = 0 \] by physical argument

\[ \Phi(x) = \begin{cases} A_1 e^{ik_1 x} + B_1 e^{-ik_1 x} & \text{, } x < 0 \\ \Phi(x) = A_2 e^{ik_2 x} & \text{, } x > 0 \end{cases} \]

**Note:** Used time dependence \[ \text{just for help} \]

Now works to make \[ \Phi(x) = \Phi(x) \]

*Their deriv \[ @ 0 \]*

\[ \therefore \text{ need it} \]

\[ B_0 \text{ at } x = 0 \]
\[ \Phi(x) = \Phi(x) \]

\[ A_1 e^0 + B_1 e^0 = A_2 e^0 \]

\[ A_1 + B_1 = A_2 \]
Next, \( \frac{d\psi_1(0)}{dx} = \frac{d\psi_2(0)}{dx} \)

\( \Rightarrow \) \( iK A_1 \bar{e} + -iK B_1 \bar{e} = iK_2 A_2 \bar{e} \)

\( iK A_1 + -iK B_1 = iK_2 A_2 \)

now \( K_1 = \sqrt{A_1} \)
\( K_2 = \sqrt{A_2} \)

The only unknowns are \( A_1, A_2 \) \& \( B_1 \).

But

Only 2 Equations!

Can't Solve all 3.

But we can do a trick that pulls out the key aspects of this problem!

\[ \begin{array}{c}
\uparrow \\
A_1 \quad \rightarrow \quad A_2 \\
\downarrow \\
\leftarrow B_1
\end{array} \]

Solve for \( A_1 \) \& \( B_1 \), keeping \( A_2 \) unknown.

In other words

\[ \text{get } A_1 \text{ \& } B_1, \text{ in terms of } A_2 \]

\( \Psi \) \( \hat{I} \) will help!
So $A_1 + B_1 = A_2$

$\exists K, A_1 - \exists K B_1 = \exists K A_2$

\[ A_1 : (1) \quad A_1 = A_2 - B_1 \]

\[ (2) \quad A_1 = \frac{A_2 (1 + \frac{K_2}{K_1})}{2} \]

Then

\[ B_1 = A_2 - A_1 \]

\[ B_1 = \frac{A_2 (1 - \frac{K_2}{K_1})}{2} \]

\[ \psi_1 (x) = \frac{A_2}{2} (1 + \frac{K_2}{K_1}) e^{K_2 x} + \frac{A_2}{2} (1 - \frac{K_2}{K_1}) e^{-\frac{K_2}{K_1} x} \quad x < 0 \]

\[ \psi_2 (x) = \frac{A_2}{2} e^{K_2 x} \quad \text{all in terms of } A_2 \]

That's it.... done..... but not too interesting.
\[ \psi_1(x) = A_1 e^{i k_1 x} + B_1 e^{-i k_1 x} \]
\[ \psi_2(x) = A_2 e^{i k_2 x} + B_2 e^{-i k_2 x} \]

To get physical, we can associate:

\[ \psi_{\text{Incident}} = A_1 e^{i k_1 x} \]
\[ \psi_{\text{Reflected}} = B_1 e^{-i k_1 x} \]
\[ \psi_{\text{Transmitted}} = A_2 e^{i k_2 x} \]

Now as \( k \)

What is chance of reflecting?

\[
\text{well } \frac{\text{Prob reflected}}{\text{Prob incident}} = \frac{\psi_R \cdot \psi_R = B_1 e^{-i k_1 x} B_1 e^{i k_1 x}}{\psi_i \cdot \psi_i = A_1 e^{i k_1 x} A_1 e^{-i k_1 x}}
\]

\[
R = \frac{B_1 \cdot B_1}{A_1 \cdot A_1} = \frac{(k_1 - k_1)^2}{(k_1 + k_1)^2}
\]

Subbing

Similarly, chance of transmitting

\[
T = \frac{\text{Prob of Trans}}{\text{Prob of Inc}} = \frac{A_2 \cdot A_2}{A_1 \cdot A_1}
\]

Now clearly \( R + T \) must = 1 probability
\[ T = 1 - \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \]

where
\[ R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \]

where
\[ k_1 = \sqrt{A_1} = \sqrt{\frac{2mE_1}{k}} \]
\[ k_2 = \sqrt{A_2} = \frac{\sqrt{2m(E-V)}}{k} \]

Rewriting
\[ R = \left( \frac{E - E - V_0}{\sqrt{E + \sqrt{E - V_0}}} \right)^2 \]

\[ T\text{ has chance of restricting back} \Rightarrow x = 0 \]

Classically, this won’t happen

Tennis ball only!

\[ \text{EXAMPLE! 4.1} \]
\[ E_{\text{tot}} = 2V_0 \text{ for} \ e^- \]
\[ R = 0.17 = 17\% \text{ chance of returning} \]
\[ @ x = 0 ! \]
The reason why is that our problem we solved is

\[ V(x) \]

\[ < \text{ step sort } \quad e \cdot x = 0 \]

\[ x = 0 \quad \frac{dV}{dx} = \infty \quad \text{not physical} \]

Real \( V(x) \) is go flat over \( \delta x \)

Now is \( \delta x > (\delta x = \frac{\hbar}{p}) \)

Then \( \delta x \) will not see \( \sqrt{\text{step}} \)

(see it has a chance to "see" the changes)

so it will not reflect as often.

Ex: 4eV for 100 eV \( e^- \)

\[ p_e = 1.2 \times 10^{-10} \text{ m} \]

\[ V(x) \] would have to go from \( V_0 \)

in \( \delta x \) size of 1 atom

Doesn't, can't happen, in macro costs very easy!
OK: we've got the prototype case! All others work the same way!

The details are in Sherrar! Here are the Solutions!

More "Traditional" scattering states

Again, 2 regions

\[ \psi'' + A \psi = 0 \]

\[ I \quad (x < 0) \quad \text{III} \quad (x \geq 0) \]

\[ E \gtrless V(x) \]

\[ \psi(x) = A e^{-kx} + B e^{kx} \quad \psi''(\infty) = A_2 e^{kx} + B_2 e^{-kx} \]

\[ k = \frac{\sqrt{2m(E-V)}}{\hbar} \]

\[ k_2 = \frac{\sqrt{2m(E-V)}}{\hbar} \]

physical

\[ A_1 \rightarrow \begin{cases} A_2 e^{kx} \quad \text{expo, blows up} \quad \text{at} \quad x = \infty \\ B_2 e^{-kx} \quad \text{goes to zero} \quad \text{at} \quad x = \text{Big} \end{cases} \]

So physically \( A_2 = 0 \)
Now when you do all the same techniques as before, you get

\[ \psi(x) = \begin{cases} \frac{B_2}{2} (1 + i\frac{k_2}{k_1}) e^{i k_2 x}, & x < 0 \\ \frac{B_2}{2} (1 - i\frac{k_2}{k_1}) e^{i k_2 x}, & x > 0 \end{cases} \]

\[ \psi(x) = \frac{B_2}{2} e^{i k_2 x}, \quad x \geq 0 \]

\[ \psi(x) = \text{incident} \]

\[ \psi(x) = \text{transmitted} \]

Note what does, say \( \psi \) reflected = \( \frac{B_2}{2} (1 - i\frac{k_2}{k_1}) e^{i k_2 x} \)

\[ \sigma = \arctan \left( \frac{k_2}{k_1} \right) \]

\[ \psi_\text{reflected} = \frac{B_2}{2} \sqrt{1 + \left( \frac{k_2}{k_1} \right)^2} e^{-i(k_1 x + \sigma)} \]

Clearly huge result -- \( \psi \) reflected \( \neq 0 \) in region \( x > 0 \) \( \Rightarrow \) 's particle can tunnel into wall

But realistically II \& III is only thick

\[ R = \frac{B_1 * B_2}{B_1 + B_2} = 1 \]

\[ T = 0 \quad \text{no tunneling} \]
However, can take this one step further and get very cool result!

\[ V = V_0 \quad \text{potential of finite width!} \]

\[ \psi = \psi_0 \]

\[ \psi'' + k^2 \psi = 0 \]

3 regions:

\[ \begin{align*}
\text{I} & \quad x < 0 \\
\text{II} & \quad 0 \leq x \leq a \\
\text{III} & \quad x > a
\end{align*} \]

\[ E_T > V(x) \quad E_T < V_0 \quad E_T > V(x) \]

\[ \psi_I(x) = \phi_1 e^{ikx} + \phi_2 e^{-ikx} \]

\[ \psi_{II} = A_1 e^{ikx} + \frac{B_1}{2} e^{-ikx} \]

\[ \psi_{III} = A_3 e^{ikx} + \frac{B_3}{2} e^{-ikx} \]

\[ k_1 = \sqrt{2mE_1/k} \quad k_2 = \sqrt{2m(E-V_0)/k} \quad k_3 = \sqrt{2mE_3/k} \]

Now meet BC's:

\[ \psi_{II}(0) = \psi_{III}(0) \quad \text{and} \quad \frac{d\psi_{II}}{dx}(a) = \frac{d\psi_{III}}{dx}(a) \]

Argue physically.
Lots of algebra but clearly

Wow! Particle Tunnelled thru
The wall ≠ Classical!

Specifically:

Tunnel prob = \frac{A_3 \cdot A_3}{A_1 \cdot A_1} = \text{TONS OF ALGEBRA}

\[
\text{UNNEC prob} = 1 + \left[ \frac{V_0^2}{4E_f N_s} \right] \sinh^2 (k_x a)
\]

Classically \( k_x = \frac{2\pi}{\lambda} = \frac{2\pi}{2\lambda} = \frac{2\pi}{4E_f N_s} \frac{1}{a} = \frac{2\pi}{2a} = \frac{\lambda}{a} \text{ small} = k_0 a \)

So \( \sinh^2 (k_x a) = \sinh^2 (\text{Big#}) \Rightarrow 0 \) so \( T = 0 \) classical
But see ε < in atom

\[ D_ε = \text{d} r \text{a atom} = A_0 \]

\[ \therefore \quad \sinh^2 (k_0 a) = \sinh^2 \left( \frac{1}{a_0} A_0 \right) \]

\[ \text{size of tunnel} = \text{atom} \sim h \]

\[ \sinh^2 (1) = \left( \frac{e + e^{-1}}{2} \right) \text{? check?} \]

**But clearly**

\[ T = \frac{1}{1 + \text{small \#}} \neq 0 \]

\[ \therefore \text{we'll quit} \]

Q.M. Tunneling!

**Example: 4.2.**

![Diagram of a wall and a baseball hitting it at an angle of 45 degrees, with a distance of 0.2 m marked and a calculation of the wave number \( k_2 \).]

\[ k_2 = \frac{2\pi}{\lambda} = \frac{2\pi}{3.8} \approx \frac{(14.45)}{(1.83)} \]

\[ = 1 \times 10^{33} \text{ m}^{-1} \]

Then \( k_2 a = 3 \times 10^{32} \)

\[ \sinh^2 (k_2 a) \approx e^{2k_2 a} \approx e^{4 \times 10^{32}} \]

\[ = 10^{32} \]

Only 17 seconds \( \approx 10^{-11} \) in lifetime of universe!
Say $\sin T \neq 0 \Rightarrow$ no need

\[ \sin h^2 (k_x a) \] reasonable \& so $k_x \approx 1$

\[ \frac{2\pi}{T} \text{ (tunnel width)} \leq 1 \]

In a way you need to be able to "see" other side

\[ \frac{1}{2} \text{ is so you wish} \text{ Tunnel} \]

\[ k \text{- decay: } \overset{\text{U}^{238}_{\text{nucleus}}}{\rightarrow} \overset{\text{He}^{2}}{\text{He}^{2}} + \overset{\text{X}^{234}_{\text{decays}}}{\text{X}^{234}} \]

\\

\[ \frac{1}{2} = \alpha \]

\[ \text{How? nucleus held together w/ STRONG NUCLEAR FORCE}! \]

\[ \text{Positrons } = e^+ \text{ particles = Residual} \]

\[ \text{So how do you end up spilling out} \]

\[ a \Rightarrow \text{nucleus?} \]

\[ \text{Tunneling!} \]
Note: For $E_{pot} = \text{Bound by Potential}$

But since $r \cong \text{tunnel width}$

you get

non zero Tunneling Probability!

Another Example: Tunnel Diode

Huge class of semiconductor Devices!
Scanning Tunneling Microscope!

The idea is, set tunnel current to say 10 nAmps

Then scan tip back & forth

across surface & move tip up & down
to maintain specified tunnel current.

In the end, you probe electron density
at atomic scale and thus "see" atoms!

Move atoms around & spelled with 23 Xenon atoms!