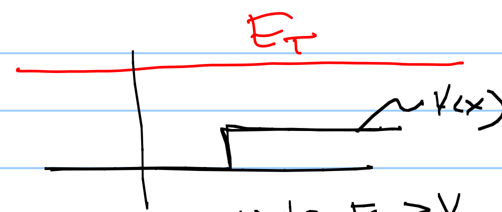
E.F. Deveney / BSC Physics PH402 Quantum #1

Note Title 4/5/2006

Scattering states
1) Free particle
2) Scattering
3) Tunneling
Scherrer chpt 4

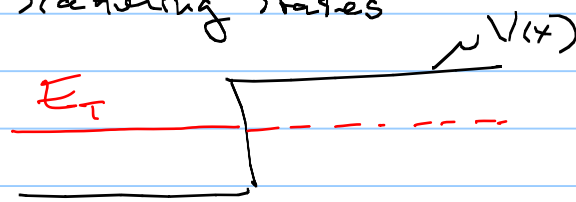
Scattering State Solns to time Indep Schrö

1) Free particle scattering states

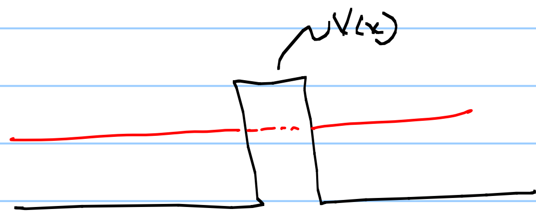


note $E_T > V$ always = free but scattering?

2.) Scattering states



3.) Then Tunneling



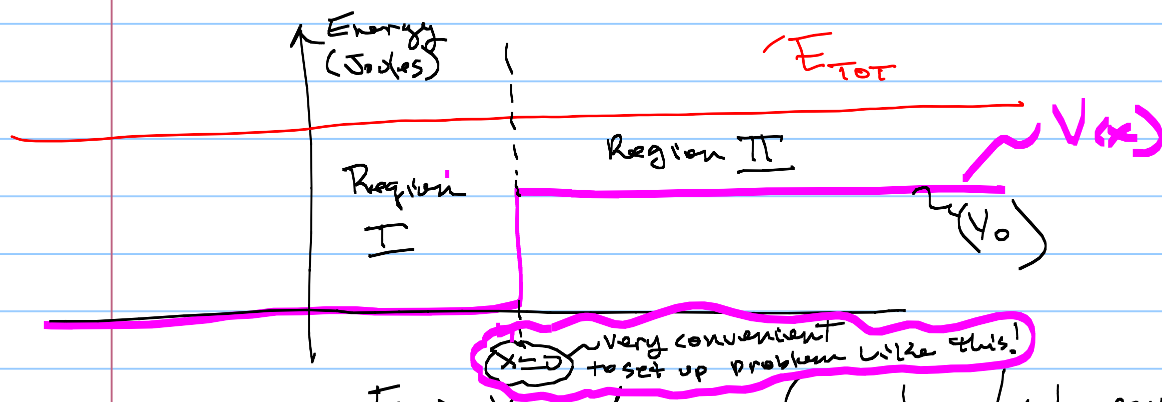
alpha-decay

tunneling microscope

tunnel diode

1) Free Particle Scattering States in Detail!

Prototype for many other solutions



$E_T > V$ always (maybe high energy states)

$$\hat{H}\psi = E_T\psi$$

$\vdots$

$$\psi'' + \underbrace{\frac{2m}{\hbar^2}(E_T - V)}_{A}\psi = 0$$

Now $E_T > V$ Always

so

$$\psi(x) = C_1 e^{i\sqrt{A}x} + C_2 e^{-i\sqrt{A}x}$$

But 2 Regions

so

use same soln in

both regions

$\underset{V=0}{\underline{\mathrm{I}}}$: $A_1 = \frac{2m}{\hbar^2}E_T$

$\underset{V=V_0}{\underline{\mathrm{II}}}$; $A_2 = \frac{2m}{\hbar^2}(E_T - V_0)$

so

$$\psi_{\mathrm{I}}(x) = A_1 e^{i(\sqrt{2mE}/\hbar)x} + B_1 e^{-i(\sqrt{2mE}/\hbar)x}$$

$$\psi_{\mathrm{II}}(x) = A_2 e^{i((\sqrt{2m(E-V_0)})/\hbar)x} + B_2 e^{-i((\sqrt{2m(E-V_0)})/\hbar)x}$$

Now to make things a bit more clear, recognize

$$\sqrt{A} = \frac{\sqrt{2mE}}{\hbar} \quad \text{or} \quad \frac{\sqrt{2m(E-V)}}{\hbar} = \frac{\sqrt{kg \cdot J}}{J \cdot s}$$

$$= \frac{\sqrt{kg}}{\sqrt{J} \cdot s} = \frac{\sqrt{kg}}{\left(\sqrt{kg \frac{m^2}{s^2}}\right) s} = \frac{\sqrt{\cancel{kg}}}{\sqrt{\cancel{kg}}\, \frac{m}{\cancel{s}} \cdot \cancel{s}} = \frac{1}{m}$$

$E_T = E_K + E_p$

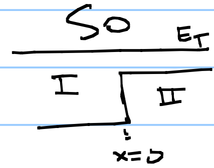
$E_T - V = \frac{p^2}{2m} \Rightarrow p = \sqrt{2m(E_T - V)}$

so since $p/\hbar = k$

$p = \hbar k$

or $\sqrt{A}$ = WAVE #! $= k = \frac{2\pi}{\lambda}$

So



I $(x<0)$ $\quad V=0$

II $(0 \le x)$ $\quad V = V_0$

$\psi_I(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}$

$\psi_{II}(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x}$

now all we have to do is make ψ well behaved for all x such that $\int_{-\infty}^{+\infty} \psi^* \psi \, dx =$ Finite

well we know soln's for $x \neq 0$ = nice ∿

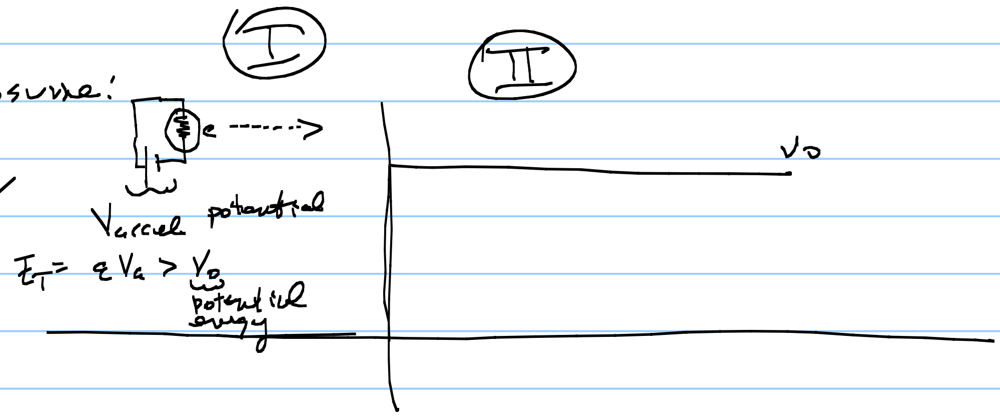
so only place we have to work hard is @ $x=0$

need $\psi_1(0) = \psi_2(0)$

$$\left.\frac{d\psi_1}{dx}\right|_{x=0} = \left.\frac{d\psi_2}{dx}\right|_{x=0}$$

Before jumping in.... try a physical argument to make life easier

assume:



Big point is that both e^- gun & free e^-'s $\Longrightarrow$

Well in region I

$$\psi(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}$$

Recall $\Psi_{(x,t)} = e^{-i\frac{E_T}{\hbar}t}\psi(x)$: recall $\underset{wave}{E_{TOT}} = \hbar\omega$

so

$$\Psi_{I}(x,t) = A_1 e^{i(k_1 x - \omega t)} + B_1 e^{+i(-k_1 x - \omega t)}$$

$$= A_1 e^{i(k_1 x - \omega t)} + B_1 e^{-i(k_1 x + \omega t)}$$

= plane travel wave $\longrightarrow$ + plane trav wave $\longleftarrow$

$\therefore$

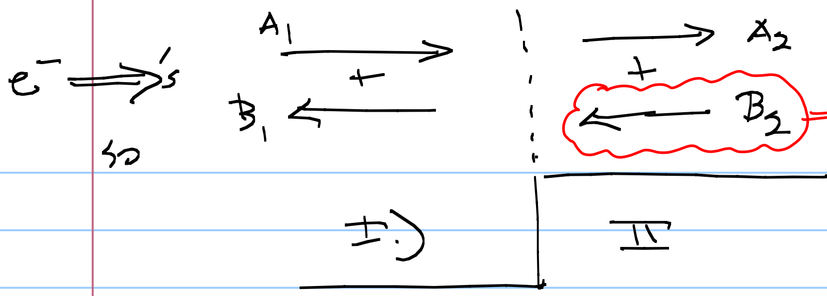
$$\Psi_{II}(x,t) = A_2 e^{i(k_2 x - \omega t)} + B_2 e^{-i(k_2 x - \omega t)}$$

$$= \longrightarrow + \longleftarrow$$

note: ω = same in I & II ....

of course $E_T = \hbar\omega$ doesn't change

But $p = \hbar k$ does... so k_1 & k_2 (ie λ_1 & λ_2)



This one doesn't make sense!

1.) in I, might start w/ $\rightarrow$
But could have reflection $\leftarrow$

2.) in II, might get transmitted $\rightarrow$
but couldn't have particles $\leftarrow$
(ie $B_2 e^{-i(k_2 x - \omega t)}$
= plane wave $\leftarrow$ starting @ $+\infty$)

So.... Make $B_2 = 0$ by physical argument

OK
$$\psi_I(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}, \quad x < 0$$
$$\psi_{II}(x) = A_2 e^{ik_2 x}, \quad x \geq 0$$

not just used time dep stuff to help argue $B_2 = 0$ now don't need it

Now Work to make $\psi_I(0) = \psi_{II}(0)$
& their derivs @ 0

So, @ $x=0$
$$\psi_I(0) = \psi_{II}(0)$$
$$A_1 e^0 + B_1 e^0 = A_2 e^0$$
$$A_1 + B_1 = A_2$$

Next, $$\frac{d\psi_I(0)}{dx} = \frac{d\psi_{II}(0)}{dx}$$

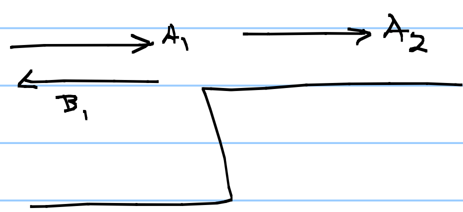
$$\Rightarrow\text{'s} \quad ik_1 A_1 e^0 + -ik_1 B_1 e^0 = ik_2 A_2 e^0$$

$$ik_1 A_1 + -ik_1 B_1 = ik_2 A_2$$

now $k_1 = \sqrt{A_1}$
$k_2 = \sqrt{A_2}$

The only unknowns are A_1, A_2 & B_1
But
Only 2 Equations!
Can't Solve all 3

BUT, we can do a trick that
pulls out the key aspects of this
problem!



Solve for A_1 & B_1, keeping A_2 unknown.
In other words

get A_1 & B_1 in terms of A_2

?
⇓

It will help!

So 1) $A_1 + B_1 = A_2$

2) $\cancel{i}K_1 A_1 - \cancel{i}K_1 B_1 = \cancel{i}K_2 A_2$

A_1: (1) $A_1 = A_2 - B_1$

(2) $B_1 = \left(\frac{K_1 A_1 - K_2 A_2}{K_1}\right)$

$$A_1 = A_2 - A_1 + \frac{K_2}{K_1} A_2$$

$$A_1 = \frac{A_2}{2}\left(1 + \frac{K_2}{K_1}\right)$$

Then

(1) $B_1 = A_2 - A_1$

(2) $A_1 = \frac{K_2 A_2 + K_1 B_1}{K_1}$

$$B_1 = A_2 - \frac{K_2}{K_1} A_2 - B_1$$

$$B_1 = \frac{A_2}{2}\left(1 - \frac{K_2}{K_1}\right)$$

So

$$\psi_I(x) = \frac{A_2}{2}\left(1 + \frac{K_2}{K_1}\right) e^{iK_1 x} + \frac{A_2}{2}\left(1 - \frac{K_2}{K_1}\right) e^{-iK_1 x}, \quad x < 0$$

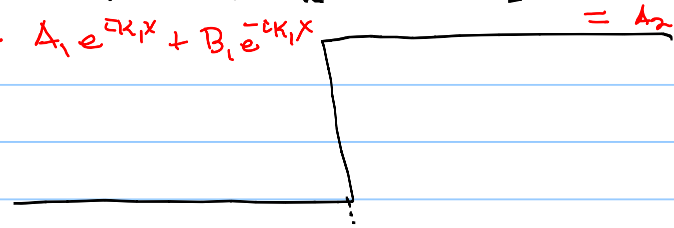
$$\psi_{II}(x) = A_2 e^{iK_2 x}$$

all in terms of A_2.

That's it.... done.... but not ~~too~~ interesting

$\psi_I(x) = \frac{A_2}{2}\left(1+\frac{k_2}{k_1}\right)e^{ikx} + \frac{A_2}{2}\left(1-\frac{k_2}{k_1}\right)e^{-ikx}$ $\qquad$ $\psi_{II}(x) = A_2\, e^{ik_2 x}$

recall $= A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}$ $\qquad$ $= A_2 e^{ik_2 x}$



$x=0$

to get physical.... we can associate

$$\psi_{Incident} = A_1 e^{ik_1 x}$$

$$\psi_{Reflected} = B_1 e^{-ik_1 x}$$

$$\psi_{Transmitted} = A_2 e^{ik_2 x}$$

Now as K

what is chance of Reflecting?

well $\frac{\text{Prob reflected}}{\text{prob incident}} = \frac{\psi_R^* \psi_R = B_1^* e^{ikx} B_1 e^{-ikx}}{\psi_I^* \psi_I = A_1^* e^{ikx} A_1 e^{-ikx}}$

$$R = \frac{B_1^* B_1}{A_1^* A_1} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}$$

subbing

Similarly; chance of transmitting

$$T = \frac{\text{prob of trans}}{\text{prob of inc}} = \frac{A_2^* A_2}{A_1^* A_1}$$

Now clearly $R+T$ must $=1$ probability

so $T = 1 - R$

$$\therefore \quad T = 1 - \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

where

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

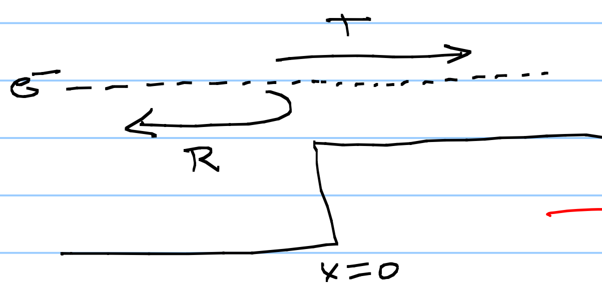
where $k_1 = \sqrt{A_1} = \sqrt{2mE_T}/\hbar$

$k_2 = \sqrt{A_2} = \sqrt{2m(E_T - V)}/\hbar$

rewriting

$$R = \left(\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}} \right)^2$$

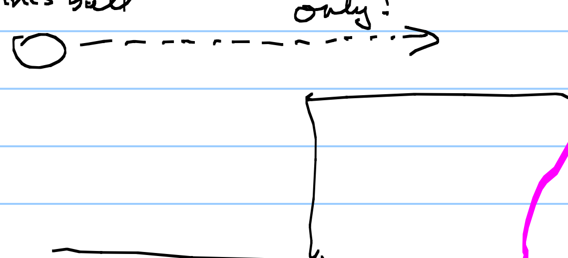
HUGE Q.M. Result!



e^- has chance of reflecting back @ $x=0$

classically this won't happen

Tennis ball only!



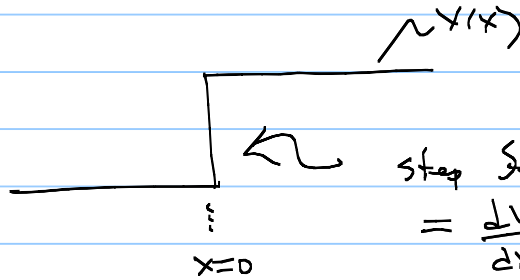
Shearer EXAMPLE! 4.1

if $E_{TOT} = 2V_0$ for e^-

$R = 0.17 = 17\%$ chance of reflecting @ $x=0$!

* Bit of reality cause really don't see 17% e^-'s doing that.

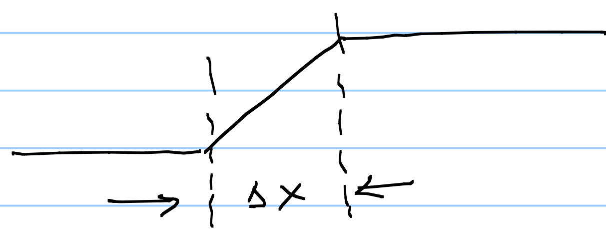
The reason why is that our problem we solved is



step funct @ $x=0$

$= \frac{dV}{dx} = \infty$ not physical

real V(x)'s go like over Δx



now is $\Delta x > \left(\lambda_e = \frac{h}{p}\right)$

then

e^- will not see (step)

(ie it has a chance to "see" the changes)

So it will not reflect as often.

Ex: 4.1 for 100 eV e^-, $\lambda_{e^-} = 1.2 \times 10^{-10}$ m

$\therefore$ V(x) would have to go from 0 to V_0

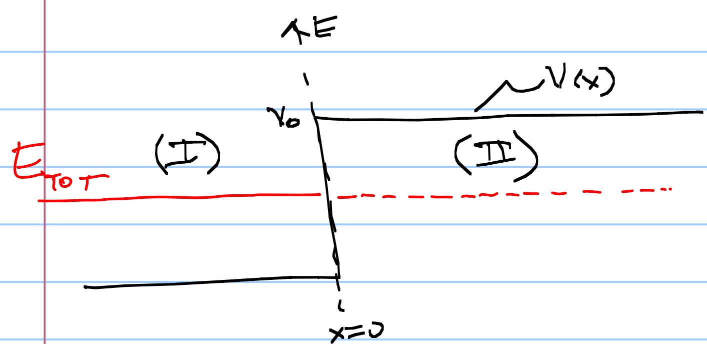
in $\Delta x <$ size of 1 atom

Doesn't, can't happen, in macro cases very easily!

OK: we've got the prototype case!
All others work the same way!

The details are in Sherrer!
Here are the Solutions!

More "Traditional" scattering states



again, 2 regions

$$\psi'' + A\psi = 0$$

I $(x<0)$ — $E_T > V(x)$

II $(x \geq 0)$ — $E_T < V(x)$

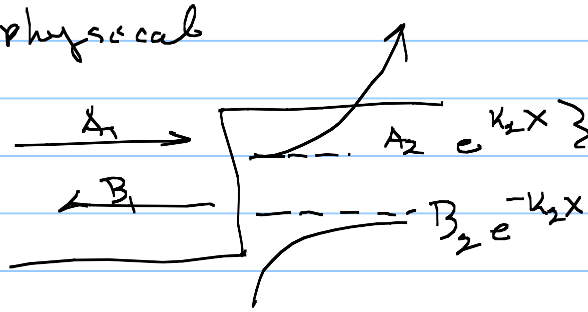
$$\psi_I(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}$$

$$k_1 = \sqrt{2mE}/\hbar$$

$$\psi_{II}(x) = A_2 e^{k_2 x} + B_2 e^{-k_2 x}$$

$$k_2 = \sqrt{2m(E_T - V)}/\hbar$$

got physical

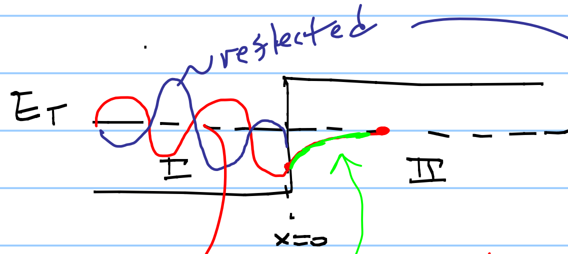


$A_2 e^{k_2 x}$ } = expo, Blows up @ x = ∞

$B_2 e^{-k_2 x}$ } = goes to zero @ x = Big

So physically $A_2 = 0$

Now when you do all the same techniques as Before, you get



Clearly Huge Result --- ψ_{e^-} & $\psi_e^* \psi_e$ in region $\neq 0$. $\Rightarrow$'s particle can Tunnel into wall

incident

$$\psi_I(x) = \frac{B_2}{2}\left(1 + i\frac{K_2}{K_1}\right)e^{iK_1 x} + \frac{B_2}{2}\left(1 - i\frac{K_2}{K_1}\right)e^{-iK_1 x} \quad ; \quad x<0$$

$$\psi_{II}(x) = B_e\, e^{-iK_2 x} \quad ; \quad x \geq 0$$

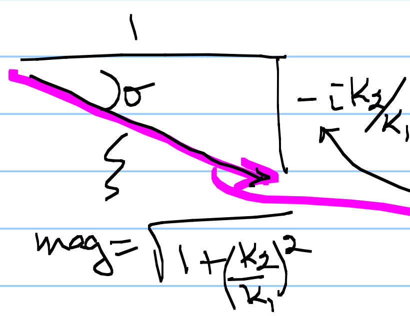
transmitted

But realistically IS II $= \infty$ly thick

$R = \frac{B_1^* B_1}{A_1^* A_1} = 1$

& $T = 0$ --- no tunneling

Note what does, say ψ_I reflected $= \underbrace{\frac{B_2}{2}\left(1 - i\frac{K_2}{K_1}\right)}_{\text{mean?}} e^{-iK_1 x}$



$\theta = \arctan \frac{K_2}{K_1}$

$\sqrt{1 + \left(\frac{K_2}{K_1}\right)^2}\, e^{-i\theta}$

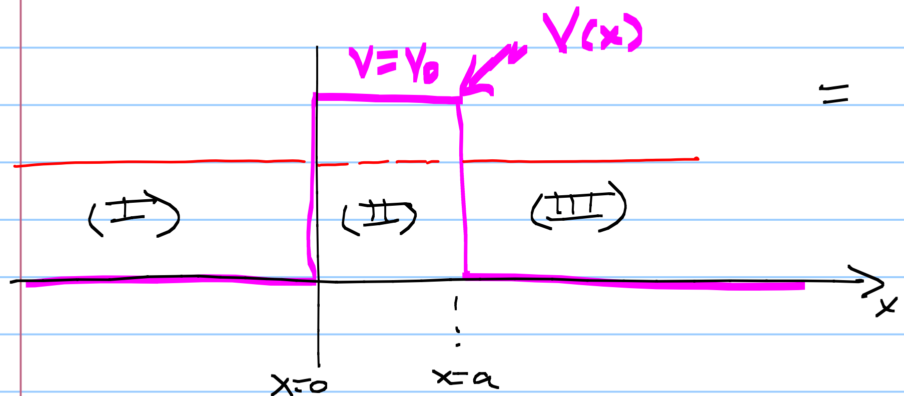
so

$$\psi_{I\,\text{Reflected}} = \frac{B_2}{2}\sqrt{1 + \left(\frac{K_2}{K_1}\right)^2}\, e^{-i\theta}\, e^{-iK_1 x}$$

$$= \frac{B_2}{2}\sqrt{1 + \left(\frac{K_2}{K_1}\right)^2}\, e^{-i(K_1 x + \theta)}$$

= phase change on reflection!

However, can take this one step further and get very cool result!



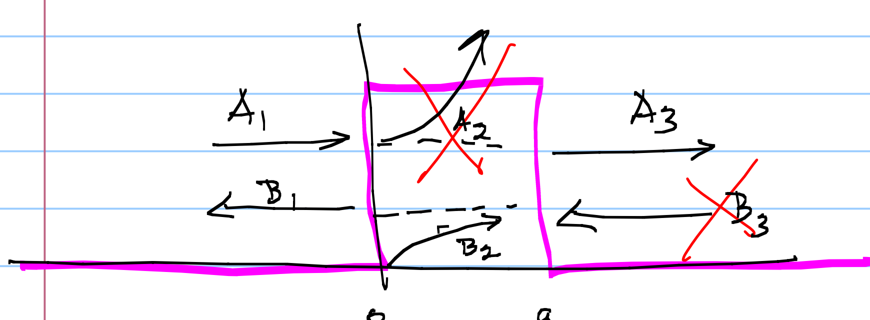
= potential of finite width!

$$\hat{H}\psi = E_T\psi$$

$$\psi'' + A\psi = 0$$

3 regions

I	II	III
$x<0$	$0 \le x \le a$	$x>a$
$E_T > V(x)$	$E_T < V_0$	$E_T > V(x)$
$\psi_I(x) = A_1 e^{ik_1x} + B_1 e^{-ik_1x}$	$\psi_{II} = A_2 e^{k_2x} + B_2 e^{-k_2x}$	$\psi_{III} = A_3 e^{ik_3x} + B_3 e^{-ik_3x}$
$k_1 = \sqrt{2mE}/\hbar$	$k_2 = \sqrt{2m(E-V)}/\hbar$	$k_3 = \sqrt{2mE}/\hbar$



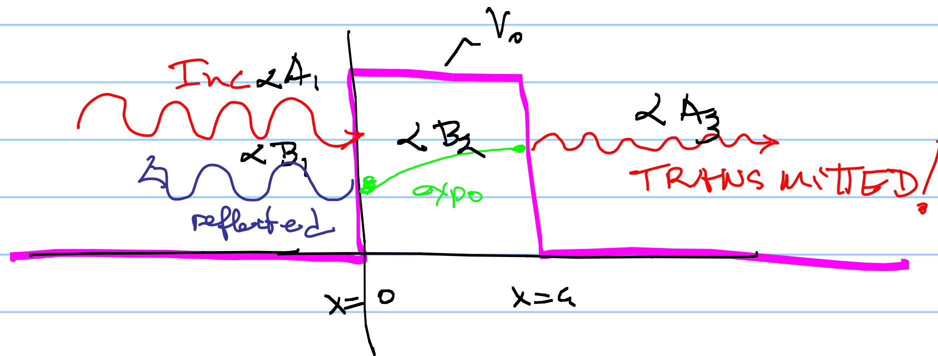
Argue physically

now meet BC's

$$\psi_I(0) = \psi_{II}(0) \quad \text{and} \quad \psi_{II}(a) = \psi_{III}(a)$$

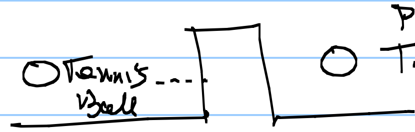
$$\frac{d\psi_I}{dx}(0) = \frac{d\psi_{II}}{dx}(0) \qquad \frac{d\psi_{II}}{dx}(a) = \frac{d\psi_{III}}{dx}(a)$$

Lots of algebra but clearly



Wow! Particle Tunnelled thru the wall

$\neq$ Classical!



prob of O Tennis Ball on other side $\neq 0$

Specifically:

$$\text{Tunnel prob} = \frac{A_3^* A_3}{A_1^* A_1} = \text{TONS of Algebra}$$

$\ast$

$$\text{TUNNEL prob} = \frac{1}{1 + \left[\frac{V_0^2}{4E_T(V_0 - E_T)}\right] \sinh^2(k_2 a)}$$

$\rightarrow a \leftarrow$

$\ast$ $\sinh^2 x = \frac{e^{2x} + e^{-2x}}{2}$ check!

$K_2 = \sqrt{2m(E_T - V_0)}/\hbar$ & a = tunnel width

Classically $K_2 = \frac{2\pi}{\lambda} = \frac{2\pi}{\lambda_{deBrog}} = \frac{2\pi}{h/p} = \frac{2\pi}{\text{small}} = \text{HUGE}$

So $\sinh^2(k_2 a) = \sinh^2(\text{BIG\#}) \Rightarrow \infty$ so $T = 0$ classically.

But for e^- in atom

$$\lambda_{e^-} \sim \text{dia atom} = a_0$$

$$\therefore \quad \sinh^2(k_2 a) = \sinh^2\left(\frac{1}{a_0} a_0\right)$$

size of tunnel = atom-ish

$$\sinh^2(1) = \left(\frac{e^{2*1} + e^{-2*1}}{2}\right) \text{ ? check ?}$$

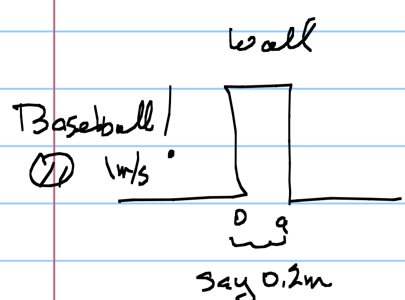
But clearly

$$T = \frac{1}{1 + \text{small \#}} \neq 0$$

$\therefore$ Will get

Q.M. tunneling!

EXAMPLE: 4.2.



$$k_2 = \frac{2\pi}{\lambda} = \frac{2\pi}{h/p} = \frac{p}{\hbar} \cong \frac{(.14\,kg)(1\,m/s)}{1 \times 10^{-34}\,J \cdot s}$$

$$= 1 \times 10^{33}\ m^{-1}$$

Then $k_2 a = 3 \times 10^{32}$

$$\sinh^2(k_2 a) \cong e^{2k_2 a} \cong e^{6 \times 10^{32}}$$

$$= 10^{10^{32}}$$

only 10^{17} seconds $\cong 10^{10^{1.1}}$ in lifetime of universe!

WO-AH

$T = \frac{1}{10^{10^{32}}} \cong 0$ to 10^{32} decimal places!

So, for $T \neq$ to $\not\cong 0$ need

$\sinh^2(K_2 a)$ reasonable & so $\underbrace{K_2 a \sim 1}$

$\uparrow$ tunnel width

or

$\left(\frac{2\pi}{\lambda}\right)$(tunnel width) $\cong 1$

or

λ of particle $\sim$ tunnel width

? where does this happen?

Atoms & Semiconductors!

in a way, you need to be able to "see" other side!

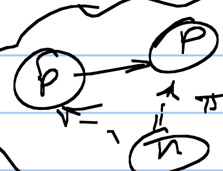
& is so you might Tunnel

ex: α-decay: U^{238} nucleus $\rightarrow \alpha + X^{234}$

${}^4_2He_2 = \alpha$

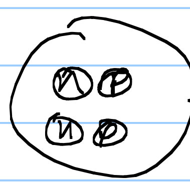
How?

nucleus Held together w/ STRONG NUCLEAR FORCE!



π ions $= q\bar{q}$ particles = Residual gluon interactions

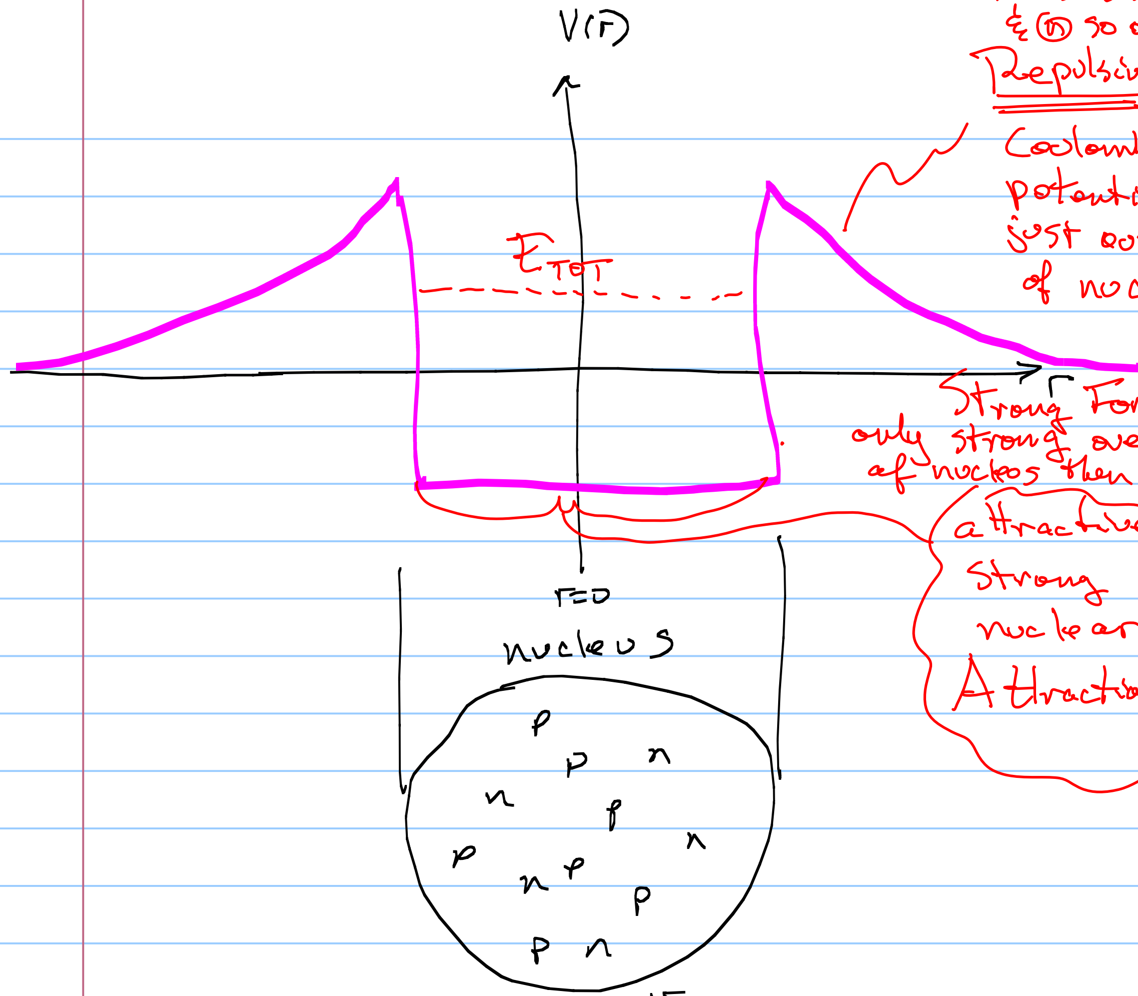
So how do you end up spitting out

a  $\Rightarrow$ nucleus?

Tunneling!

nuc = Bunch of ⊕p & ⓝ so overall
Repulsive

Coolombic potential just ouside of nucleus!



Strong Force is only strong over scale of nucleus then drops off quickly!

attractive strong nuclear Attraction!

$\cong 10^{-15} m = \text{Fermi}$

Note: For E_{TOT} = Bound by Potential
But
since $\lambda \cong \underbrace{\text{atom}}_{\text{tunnel width}}$

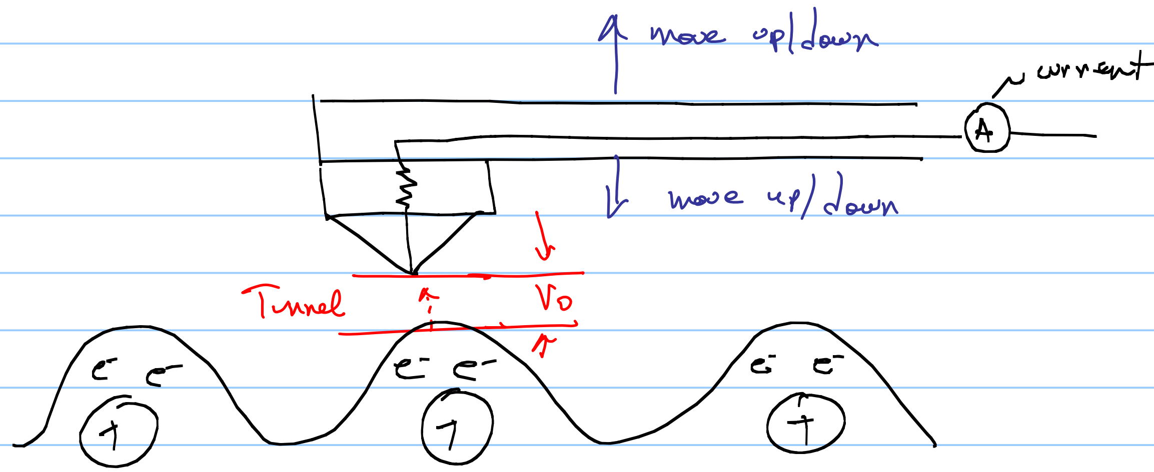
you get non Zero Tunneling Probability!

Another Example: Tunnel Diode

Huge class of semiconductor Devices!

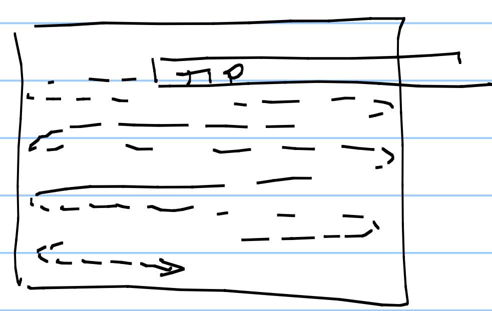
Another Example: $\Longrightarrow$

Scanning Tunneling Microscope!



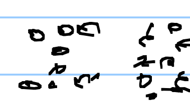
idea: is, set tunnel current ~ say 10 nAmps

Then scan tip back & forth



across surface & move Tip up & down to maintain specific tunnel current.

In the end, you probe e⁻ density @ atomic scale and thus "see" atoms!

Move atoms around & Spelled  M w/ 23 Xenon atoms!