Harmonic Oscillator $\Rightarrow$ Springs

all systems near Stable equilibrium!

stable equi small disturbances; $x_0$ = equilibium point.

unstable equi catastrophic displacement from that equilibium. unstable:

Stable Equilibrium Potential:

The potential energy of a spring is $\frac{1}{2}kx^2$.
Thus classical is ———

If you see stable equilibrium — anywhere!

Then you immediately think of Springs — harmonic oscillators.

D) Restoring Spring Forces

(H) Hard spring potential energies.

Thus when you "see" stable equlibrium in Q.M. problems you

\[ \frac{\hota}{\hota t} = \frac{\Delta}{\hota} \]

energy eigenstates so

\[ \Delta = e^{i \frac{\Delta}{\hota} \hota} (x) \]

\[ \Delta \psi = \hbar \psi \]

Note guess it will be quantized \( \hbar \)’s w/ corresponding eigenfunction.

\[ \left( \frac{\hota^2}{2m} + \frac{\hbar^2}{2m} \right) \psi_n = \hbar \psi_n \]

\[ \frac{-\hbar^2}{2m} \frac{\hbar^2}{2m} \psi_n + \frac{\hbar^2}{2m} \psi_n = \hbar \psi_n \]

= time indep Schröd. For all harmonic oscillators!

So need to be able to solve...
First... examples!

- Concentrate on phase 3.

If you push \( \pm 2 \) → a bit, it's the e−e− interaction & pushes

while on the other side you make \( \pm 1 \) more (+) thus attraction.

Thus carbon \( \pm 2 \) upon displacement from equilibrium is restored by oscillate about its original equilibrium position.

So, think springs, \( E_p = \frac{1}{2} k x^2 \) model that works well!
later on, Townsend Chapt 14, you get a
big surprise when you look &

\[ \mathcal{E}_{\text{kin}} = \frac{1}{2} \left( E^2 + \frac{1}{x^2} \right) = \frac{\text{energy}}{\text{vol}} \]

That the \( \mathcal{E}_{\text{kin}} \) also looks like
Harmonic Oscillators
ie phonons!

Also: more complicated potentials can be model to 2nd order as Harmonic
Oscillators:

\[ V(x) = k x^2 \]

to see this; of course take the Taylor series

expansion of \( V_{\text{model}}(x) \) about \( x_0 \)

\[ V_g(x) = V_g(x_0) + \frac{dV_g}{dx} |_{x_0} x + \frac{1}{2} \frac{d^2V_g}{dx^2} |_{x_0} x^2 + \text{H.O.T.} \]

What is really significant is not \( |V_g(x)| \) but
\[ \Delta V_g(x) \]

ie potential differences!

So FREE to call this \( \text{potential} = 0 \)

\[ \text{ex: Earth is ground (\text{\#}) = electrical potential} = 0 \text{ by definition} \]
So... solve Harmonic Oscillator Potential in QM.

\[-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} k x^2 \psi = E \psi \]

(well put n's in \( \psi \) and)

I am going to follow Griffith’s Intro to QM. Chpt 2, pg 32.

For entire Harmonic Oscillator.

So will recall... \( \Sigma F = m \ddot{x} \)

\[-kx = m \ddot{x} \]

\[\ddot{x} + \omega_0^2 x = 0\]

\[\omega_0 = \sqrt{\frac{k}{m}}\]

so \( k = m \omega_0^2 \)

\[-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} m \omega_0^2 x^2 \psi = E \psi\]

Want to solve this!

2-ways

1) Clever Algebraic Method
called ladder \( \delta \)'s

2) Brute Force
Solve this...

Of course
results will
be the same!
D) Algebraic soln to Harmonic Osc time indep soln

\[
-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi = E \psi
\]

Also, rewrite (just clear, Dirac did same thing with \( E \) for field)

\[
\frac{1}{2m} \left[ \left( \frac{d}{dx} \right)^2 + (m\omega x)^2 \right] \psi = E \psi
\]

Now there are more \( \delta \)'s

\( \delta \) will have to be a bit careful

Idea: say \( u = i\nu \)

Then

\[
(u - i\nu)(u + i\nu) = u^2 + i\nu - i\nu + \nu^2
\]

Notice I was careful with the order here.

Old so keeping in "mind" order is important, hence for \( \nu \neq 4 = \# \delta \)

Then

\[
(u - i\nu)(u + i\nu) = u^2 + \nu^2
\]

Looks a awful lot like

Thus ... is we can and then we might be able to rewrite Schröd \( \psi \) as \( (u - i\nu)(u + i\nu) = 0 \)

\( \frac{1}{2} \) solution for each of the multi-valutive sectors.
OK: Try this: \[ \text{Hint (yeah it works!)} \]

\[ A_\pm = \frac{1}{\sqrt{2m}} \left( \frac{\hbar}{i} \frac{d}{dx} \pm i m v x \right) \Rightarrow \text{ie our } u \mp iv \]

Suggested by look of Schröd

\[ \frac{1}{2m} \int \left( \frac{\hbar}{i} \frac{d}{dx} \right)^2 + (i m v x)^2 \psi = E \psi \]

Now what? Remember we assumed

\[ iuv - ivu = 0 \quad \text{so } u \psi v = \psi \]

Here

\[ u \psi v \neq \psi \text{ but } 0 \text{'s} \]

so Test to see if This is True if we will get a SURPRISE!

Look, \( u = 3, v = 2 \Rightarrow i(3)(2) - i(2)(3) = 0 \)

No surprise for ordinary \( \# \)s

Could rewrite as

\[ \delta [u, v] = \delta (uv - vu) = 0 \]

\[ [u, v] = (uv - vu) \]

called Commutation relation!

\[ \text{How they commute!} \]

But for \( u \psi v \neq \psi \text{ but } 0 \text{'s} \)

The \[ [u, v] = 0 \]

\text{ie you say: } u \& v \text{ ordinary } \# \text{ commute with each other.}
So here we have
\[ \hat{A}_+ = \frac{1}{\sqrt{2m}} \left( \frac{\hbar}{i} \frac{d}{dx} + imwx \right) \]

You can ask a silly question do \( \hat{A}_+ \) & \( \hat{A}_- \) commute?

\[ [\hat{A}_+, \hat{A}_-] = \hat{A}_- \hat{A}_+ - \hat{A}_+ \hat{A}_- \]

\[ [\hat{A}_-, \hat{A}_+] = \chi w \]

So indeed \( \chi \) is not same as ordinary \( \chi \)

But that is not the "Big" surprise here

at least motivated to look at

\[ \hat{A}_- \hat{A}_+ \text{ and see what that is} \ldots \text{to see, must try a test function} \]

\[ \text{see what happens} \ldots \text{here is where the surprise comes}! \]

\[ \hat{A}_- \hat{A}_+ \psi(x) = \frac{1}{2m} \left( \frac{\hbar}{i} \frac{d}{dx} - imwx \right) \left( \frac{\hbar}{i} \frac{d}{dx} + imwx \right) \psi(x) \]

Test function

\[ = \frac{1}{2m} \left( \frac{\hbar}{i} \frac{d}{dx} + imwx \right) \left( \frac{\hbar}{i} \frac{d}{dx} - imwx \right) \psi(x) \]

\[ = \frac{1}{2m} \left[ -\frac{\hbar^2}{d^2x} - imwx \frac{d}{dx} \right] \psi(x) \]

Note care in

\[ \text{DERB} \]

\[ \frac{1}{2m} \left[ -\frac{\hbar^2}{d^2x} + imwx \frac{d}{dx} \right] \psi(x) \]

\[ = \frac{1}{2m} \left[ \frac{\hbar^2}{d^2x} + imwx \frac{d}{dx} \right] \psi(x) \]
\[(\hat{A} - \hat{a}_+ - \frac{1}{2} \hbar \omega) \psi = E \psi = \text{New}\]

Note: \(\psi(x)\) is general and now can be removed.

\[\hat{A} - \hat{a}_+ = \frac{1}{2m} \left[ \frac{\hbar}{i} \frac{d}{dx} \right]^2 + \left( m \omega x \right)^2 + \frac{\hbar \omega}{2} \]

Who cares still?

\[\left( \hat{A} - \hat{a}_+ - \frac{1}{2} \hbar \omega \right) \psi = \frac{1}{2m} \left[ \left( \frac{\hbar}{i} \frac{d}{dx} \right)^2 + \left( m \omega x \right)^2 \right] \psi \]

\[\hat{A} \psi = E \psi \quad \text{Harm-Osc}\]

\[\frac{1}{2m} \left[ \left( \frac{\hbar}{i} \frac{d}{dx} \right)^2 + \left( m \omega x \right)^2 \right] \psi = E \psi\]
\[ \left( \hat{a} - \hat{a}^* \right) \psi = E \psi = \text{New} \]

Schroed Equation for All Oscillators!

If you're clever, you'll ask why not have started from

\[ \hat{a} \psi \?
\]

and you have gotten to

\[ \left( \hat{a} + \hat{a}^* \right) \psi = E \psi \Rightarrow \text{Also an equivalent representation of Schröd equation for Harmonic Oscillator.} \]

\[ \hat{a} - \hat{a}^* \frac{\hbar \omega}{2} = \hat{a} + \hat{a}^* \frac{\hbar \omega}{2} = \text{Harmonic Oscillator} \]

so

\[ \text{OK, now what?} \]
What have we done?

\[ H_{H_0}, \Psi = E \Psi \]

Replaced \( \frac{1}{2m} \left[ \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 + (m\omega x)^2 \right] \Psi = E \Psi \)

\[ \begin{align*}
\hat{H}_{H_0} \Psi &= E \Psi \\
\left[ \hat{a}_+ \hat{a}_- + \frac{\hbar \omega}{2} \right] \Psi &= E \Psi \\
\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + m\omega^2 x^2 \Psi &= E \Psi
\end{align*} \]

OK so how has this algebraic gymnastics helped us?

Well to appreciate answer to that we must work thru one more exercise.

Watch this....

\[ \hat{H}_{H_0} \Psi = E \Psi \]

Try \( \Psi \Rightarrow \hat{a}_+ \Psi \)

\[ \begin{align*}
\hat{H}_{H_0} \left( \hat{a}_+ \Psi \right) &= E \left( \hat{a}_+ \Psi \right) \\
\text{is it still an eigenvalue problem?}
\end{align*} \]

Use new \( \hat{a}_+ \)

\[ \begin{align*}
\left[ \hat{a}_+ \hat{a}_- + \frac{\hbar \omega}{2} \right] \hat{a}_+ \Psi &= E \hat{a}_+ \Psi \\
(\hat{a}_+ \hat{a}_- + \frac{\hbar \omega}{2} \hat{a}_+ \Psi) &= E \hat{a}_+ \Psi \\
= \hat{a}_+ \left( \hat{a}_- \Psi + \frac{\hbar \omega}{2} \Psi \right) &= E \hat{a}_+ \Psi
\end{align*} \]
\[
a_+ \left[ (a-a_+-\frac{\hbar \omega}{2}) \psi + \hbar \omega \psi \right]^2 = E_{q+}(a_+ \psi)
\]

1. Wait! Recognize this is our new spigg \[ E_\psi \]
2. \[ a_+ \left[ E_+ \psi + \hbar \omega \psi \right] = E_{q+}(a_+ \psi) \]
3. \[(E_++\hbar \omega)(a_+ \psi) = E_{q+}(a_+ \psi) \]

For \[ \gamma = q_+ \gamma \]

What have we done?

- We got an eigenvalue problem back!

If we asked \[ \hat{H} (a_+ \psi) = E_{q-}(a_+ \psi) \]

and done the same procedure yield

\[ (E_+-\hbar \omega)(a_+ \psi) = E_{q-}(a_+ \psi) \]

So let's recap:

\[ \gamma = a_+ \psi \]

\[ E_{q-} = E_+ - \hbar \omega \]
The harmonic oscillator potential:

\[ H \psi = E \psi \]

\[ \frac{-\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi = E \psi \]

\( \omega = \frac{\sqrt{k}}{m} \)

Define

\[ a_+ = \frac{1}{\sqrt{2m}} \left( \frac{\hbar}{i} \frac{d}{dx} - im \omega x \right) \]

Then

\[ H \psi = E \psi \]

\[ (a_+ a_- - \frac{1}{2} \hbar \omega) \psi = E \psi \]

and

\[ (E + \hbar \omega)(a_+ \psi) = E a_+ (a_+ \psi) \]

\[ (E - \hbar \omega)(a_- \psi) = E a_- (a_- \psi) \]

So: if you know \( \psi_i \in E_i \), say

Then

\[ (E_1 + \hbar \omega)(a_+ \psi_i) = E_{a_+}(a_+ \psi_i) \]

so apparently \( a_+ \psi_i \) is raising \( \psi_i \) to next energy state \( \psi_2 \)

\[ (E_1 + \hbar \omega) \psi_2 = E_2 \psi_2 \]

\[ \frac{1}{2} E_2 = E_1 + \hbar \omega \]
\( \frac{1}{2} \) could also do

\[
(E_2 + \hbar \omega)(\hat{a}^\dagger \psi_2) = E_{a^+} (\hat{a}^\dagger \psi_2)
\]

or

\[
(E_2 + \hbar \omega) \psi_3 = E_3 \psi_3
\]

so \( \psi_3 = \hat{a}^\dagger \psi_2 \)

\[
\frac{1}{2} E_3 = E_2 + \hbar \omega
\]

\( \hat{a}_+ = \text{raising } \hat{a} \)

Similarly, say start at \( \psi_3 \)

\[
(E_3 - \hbar \omega)(\hat{a}_- \psi_3) = E_{a} (\hat{a}_- \psi_3)
\]

\( \hat{a}_- \psi_3 \text{ lowers state} \)

\[
(E_3 - \hbar \omega) \psi_2 = E_2 \psi_2
\]

so \( \psi_2 = \hat{a}_- \psi_3 \)

\[
\frac{1}{2} E_3 = E_3 - \hbar \omega
\]

\[ \text{YEAH!} \]

That's our TRICK

\( \hat{a}_+ = \text{raising } \hat{a} \); \( \hat{a}_+ \psi_n = \psi_{n+1} \Rightarrow E_{n+1} = E_n + \hbar \omega \)

\( \hat{a}_- = \text{lowering } \hat{a} \); \( \hat{a}_- \psi_n = \psi_{n-1} \Rightarrow E_{n-1} = E_n - \hbar \omega \)
Could we even do anything useful?

Remember, we are trying to solve the Harmonic Oscillator problem. Is there were an easier way, we'd do it. Also, this raising & lowering & formalism is how Q.E.T. is done in a Fock space!

\[ a_+ a_- |0\rangle = a_+ |1\rangle = \uparrow \]

Enough. Let's do one of our problems w/ this Technique.

To do so, we need to realize 1 more thing:

Say you have \( \Psi_n \)

Then lower it \( a_- \Psi = \Psi_{n-1} \)

Then again \( a_- \Psi_{n-1} = \Psi_{n-2} \)

& again

\[ \vdots \]

\[ \Psi \]

\[ \text{How long can you do this?} \]

Clearly, at least in Schrödinger non-relativistic equation, there is a bottom

\[ E_i \]

\[ \Psi_i \]

\[ \text{one less: } a_- \Psi_i = \Psi_{i-1} = 0 \]

\* Q.E.T. based on Dirac equal & neg. energies must reconcile. This argument by definition (??) makes the vacuum = 0 & then all other energies are above 0.
So, at some point when lowering states, call this lowest norm $\Psi_0$ state, not the $\Psi$ we squarewell, we do something with that?

\[
\frac{1}{\sqrt{2m}} \left( \frac{\hbar}{i} \frac{d}{dx} - i m \omega x \right) \Psi_0 = 0
\]

\[
\frac{1}{\sqrt{2m}} \left( \frac{\hbar}{i} \frac{d}{dx} \Psi_0 - i m \omega x \Psi_0 \right) = 0
\]

\[
\frac{d\Psi_0}{dx} = -\frac{m \omega}{\hbar} \Psi_0
\]

\[
\int \frac{d\Psi_0}{\Psi_0} = \int \left( -\frac{m \omega}{\hbar} \right) \Psi_0 dx
\]

\[
\ln \Psi_0 = -\frac{m \omega x^2}{2\hbar} + C
\]

\[
\Psi_0(x) = e^{C} e^{-\frac{m \omega x^2}{2\hbar}}
\]

\[
\Psi_0(x) = A e^{-\frac{m \omega x^2}{2\hbar}}
\]

Cool!
\[ \psi_0 = A_0 e^{-\frac{\hbar \omega}{2} x^2} = \text{ground state H.O.} \]

what is its energy?

well \[ H \psi_0 = E_0 \psi_0 \]

\[ (a_+ a - \frac{\hbar \omega}{2}) \psi_0 = E_0 \psi_0 \]

so \[ E_0 = \frac{\hbar \omega}{2} = \text{ground energy of H. Osc.} \]

not ground lowest energy is NOT zero!

Classical Hamilton can be still but not true in Q.M.

idea is related to uncertainty principle!

Conclude: All Harmonic Oscillators have energy always non-zero.

The vacuum of H.Osc in ground state is loaded with energy.

> Back to us, we \( \psi_0 \) is now \( E_0 \)

so can get all \( \psi_n \) \& \( E_n \) by using our

\[ \psi_n (x) = A_n (a_+ \text{)}^n e^{-\frac{\hbar \omega}{2} x^2} \]

\[ \xi \ E_n = (n + \frac{1}{2}) \frac{\hbar \omega}{2} \]

**WOW! Most ELEGANT!**

Note: \( \xi \) in only to \( \psi_n \) Normaliz constant.
Example:

What are $\psi, \xi, E$ of a H. Osc?

$\psi_i = A_i \alpha \psi_0$

$= \frac{A_i}{\sqrt{2m}} \left( \begin{array}{c} \frac{x}{i} \frac{d}{dx} + \text{i} \mu \omega \chi \end{array} \right) \left[ e^{-\frac{\mu \omega x^2}{2m}} \right]_{\psi_0}$

$= \frac{A_i}{\sqrt{2m}} \left[ \begin{array}{c} \frac{x}{i} \left( \frac{-\mu \omega x^2}{2m} \right) x^2 + \text{i} \mu \omega x e^{-\frac{\mu \omega x^2}{2m}} \end{array} \right]$

$= \left( iA_i \omega \sqrt{2m} \right) x \chi e^{-\frac{\mu \omega x^2}{2m}}$

Recall: observables = $\langle \xi_i \rangle$

$\xi \neq \psi^* \psi$ so it's cancel

$\frac{1}{2} E_1 = (1 + \frac{1}{2}) \mu \omega = \frac{3}{2} \mu \omega$

Most elegant...;''

gen $\psi_{100}(x) \not\in E_{100}$ is

$\psi_{100}$ like!
OK: That was elegant, sophisticated
\( E\Phi \) you will use it again and again
\( \text{E} \) Field
\( \text{Q} \) F.T.

But could have just solved

\[
\hat{H}\Psi = E\Psi
\]

\[
-\frac{x^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}mv^2 x^2 \Psi = E\Psi
\]

The Brute Force way!

Not Fun --- See Scharmer, see Cooper.
Many Tricks
Not so elegant
doesn't lend itself in general situations

must define \( E = \sqrt{\frac{2m}{\hbar}} x \)

rewrite Schröd in term of deriv of \( \frac{d}{dx} \)

\[
\frac{1}{\hbar^2} \frac{\partial^2 \Psi}{\partial E^2} = (E^2 - x) \Psi \text{ do some magic}\]

\[
\Psi_n(x) = \left( \frac{\sinh}{\pi n!} \right)^{\frac{1}{4}} \frac{1}{2^n n!} H_n(E) e^{-\frac{E^2}{2}}
\]

\( \text{This is normalized} \)

\text{Thats good}

\text{and so on,}

\( H_0(x) = 1 \)
\( H_1(x) = 2x \)
\( H_2(x) = 4x^2 - 2 \)
\( H_3(x) = 8x^3 - 12x \)
How did

Raising-Lowering Technique = Brute Force?

\[ \psi_0 = A_0 e^{-\frac{m_0 v^2 x^2}{2k_0}} = (\frac{m_0 v^2}{\pi k_0})^{\frac{1}{4}} \frac{1}{\sqrt{2}v^2} \psi_0 (x) e^{-\frac{x^2}{2}} \]

\[ k_0 = \frac{m_0 v^2}{\hbar} \]

\[ \sqrt{x} \]

Yep

O-k Huce

part of this form so that it is

Normalized!
So, what do \[ \frac{-\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} m \omega^2 x^2 \psi = E \psi \]

\[ \omega = \sqrt{\frac{k}{m}} \]

\[ \psi_n(x) = A_n(q_{n}) e^{\frac{-m \omega x^2}{2\hbar}} ; E_n = (n + \frac{1}{2})\hbar \omega \]

Selections look like?

\[ \psi_0(x) \]
\[ \psi_1(x) \]
\[ \psi_2(x) \]
\[ \psi_3(x) \]

Then

\[ \frac{\psi_{100}}{\psi_{100}} \]

*Note* \( n \to \infty \) looks a bit like classical

\[ \psi_{100} \] w/ prob of finding particle @

\[ \frac{\psi_{100}}{\psi_{100}} \]

w/ max
Griffith's notes:

1) Q.M. Harmonic oscillator also can penetrate beyond its classical limits (i.e. range 5 wants energy)

2) Weird Set

\[ \mu = \text{odd} \]

\[ \psi_n^* \psi_n = 0 \]

Want even spin

Q.M. H. Osc on the Center!

3) Only when \( n > 0 \) do you start seeing classical resemblance?

E.J. Dowry: H. Osc from Griffiths
Intro to Q.M.
So is you home

\[ \hat{H} \Psi_n = E_n \Psi_n \]

\[ \left( \frac{p^2}{2m} + \frac{1}{2} k x^2 \right) \Psi_n = E_n \Psi_n \]

\( \Psi_n \) are Hermite Polynomials

and you got energies

\[ E_n = (n + \frac{1}{2}) \hbar \omega \]

* Energy can't = 0! (i.e. it would violate uncertainty relation)

\[ E_{\text{lowest}} = \left( 0 + \frac{1}{2} \right) \hbar \omega \]

\[ = \frac{\hbar \omega}{2} \]

\[ I \xi = 0 \] exactly

\[ \Rightarrow x = 0 \]

\[ \text{You'd think this is zero in energy because} \]
\[ \text{The string is unstretched} \]

Classically, lowest energy is \( x = 0 \)

But is \( x = 0 \)

\[ \text{exactly, } \Delta x = 0 \]

Then \( E_p \) must be \( \frac{\hbar}{\Delta x} \)

Not zero.

So by the uncertainty principle, you cannot have

Zero -- min energy of Harm Oscill.
\( E_{n} = (n + \frac{1}{2}) \hbar \omega \) 

Note: the energy comes in

Discrete sums = \( \hbar \omega \)

\( \frac{1}{2} \hbar \omega \), \( n = 0 \) \( \psi_0 \)

\( 1 \frac{1}{2} \hbar \omega \), \( n = 1 \) \( \psi_1 \Rightarrow \hbar \omega \)

\( 2 \frac{1}{2} \hbar \omega \), \( n = 2 \) \( \psi_2 \Rightarrow 2 \hbar \omega \)

\( \vdots \)

\( n \frac{1}{2} \hbar \omega \), \( n \geq 0 \) \( \psi_n \Rightarrow n \hbar \omega \)

There quantized energies:

Characteristic of all harmonic oscillators:

Ex: Townsend Ch 7 § 14

\[ \hat{H} = \frac{1}{8\pi} \int d^3 \mathbf{r} \left[ \varepsilon_0 E^2 + \frac{1}{\varepsilon_0} B^2 \right] \]

\[ = (\text{DIRAC's equation}) \]

\[ = \sum_{k, \tau} \hbar \omega \left( a_{k,\tau}^+ a_{k,\tau} + \frac{1}{2} \right) \]

\[ = \text{exactly the same as} \]

\[ \text{The} \ \hat{\mathbf{H}} \ \text{using ladder ops!} \]

\( \text{Harmonic oscillator} \)
Therefore, $E_{\frac{1}{2}M} = \text{Harmonic oscillators characterized by energies }$

$$(n + \frac{1}{2})\hbar\omega$$

$n = 0 \quad E = \frac{\hbar\omega}{2} \quad \text{no zero, zero point energy}$

$n = 1 \quad (1 + \frac{1}{2})\hbar\omega$

$n = 2 \quad (2 + \frac{1}{2})\hbar\omega$

Field!

The vacuum is not Boring! $n = \# \text{ of quanta}$ of energy $\pmb{\hbar\omega}$

In fact, it has $\frac{\hbar\omega}{2}$ for each $n$ running to $\infty$

"Vacuum = \infty energy in its ground state!"

Similarly
Solids clearly:

\[ E_{\text{solid}} = E_{\text{oscill}} \]

Thus \[ E_n = (n + \frac{1}{2}) \hbar \omega \]

These discrete energy jumps

\[ = \text{phonons} \]