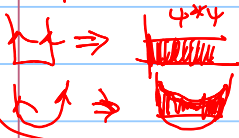
E.F. Deveny / BSC Physics Quantum PH 402: The Harmonic Oscillator Schrö time indep soln.

Note Title 7/2006

* Keep in mind: Square well potential



$\psi^*\psi$ ≅ constant, here

$\psi^*\psi$ = Bigger near end pts

Refs: Scherrer hot more

D. Griffiths: Intro to Q.M.

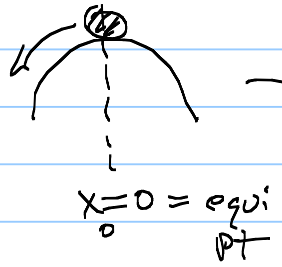
Harmonic Oscillator ⇄ Springs

all systems near stable equilibrium!

stable equi

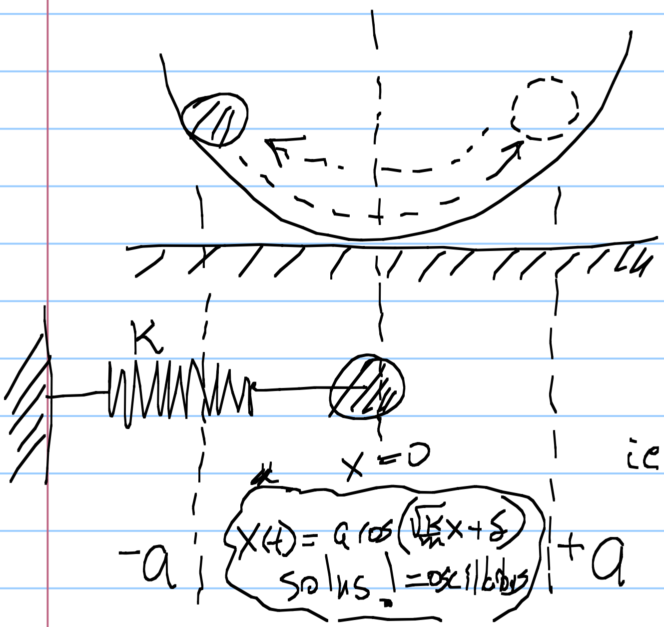
small displacements, disturbances, from equi. cause oscillations about that equi pt.

unstable equi



unstable: catastrophic displacement from that equi:

Stable Equilibrium Potential:



$x(t) = a\cos\left(\sqrt{\frac{k}{m}}x+\delta\right)$ solns! = oscillations

project ball in bowl position onto x-axis

→ There motion looks like m attached to spring:

ie a HARMONIC OSCILLATOR

The potential energy of a spring is: $\frac{1}{2}kx^2$

Thus claim is -- ....

IF you see stable equilibrium --- anywhere!

Then you immediately think of Springs, Harmonic Oscil's

1) restoring Spring Forces
2) $\frac{1}{2}kx^2$ spring potential energies.

Thus when you "see" stable equi in Q.M. problems you

$$\hat{H}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

energy eigenstates so

$$\Psi = e^{i\frac{E_n}{\hbar}t}\psi(x)$$

$$\hat{H}\psi_n = E_n\psi_n$$

⟸ note guess it will be quantized E_n's w/ corresponding eigen function

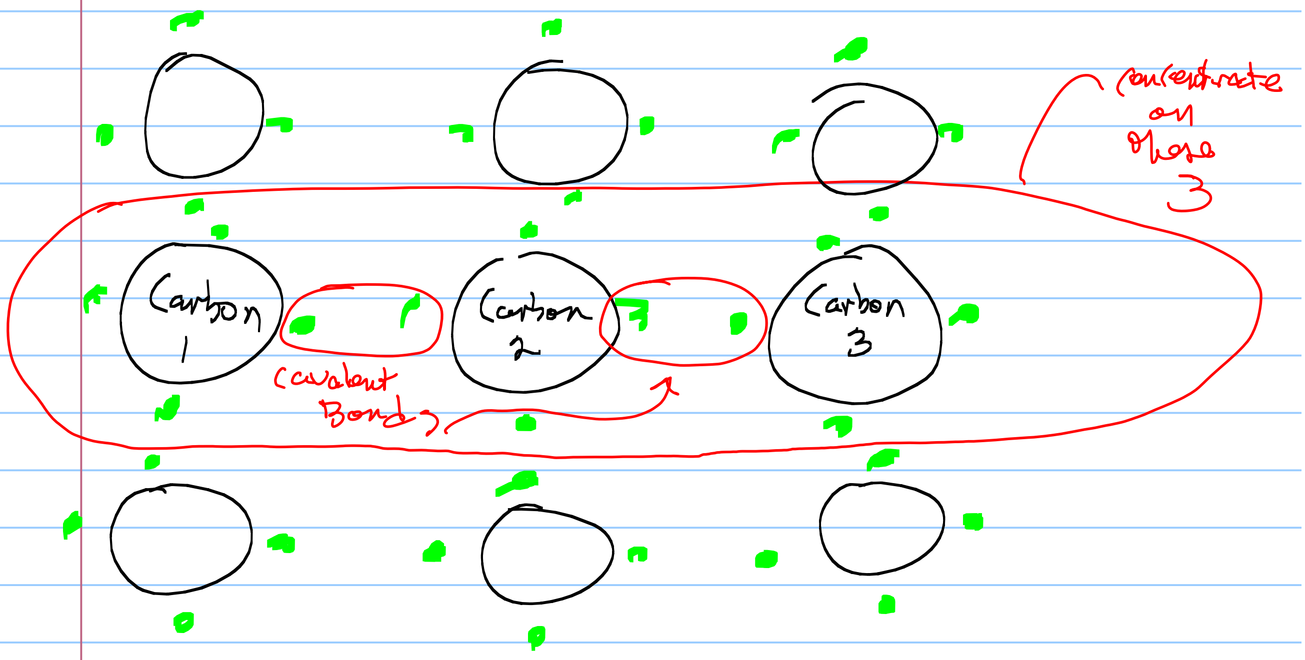
$$\left(\frac{p^2}{2m} + E_{p\,\text{spring}}\right)\psi_n = E_n\psi_n$$

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} + \frac{1}{2}kx^2\psi_n = E_n\psi_n}$$

= time indep Schrö For all Harmonic Oscillators!

so, need to be able to solve!

FIRST.... examples!

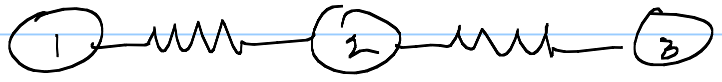


If you push #2 → a bit, it ↑'s the $e^- - e^-$ interaction & pushes ←

while on the other side you make #1 more (+) thus ↑ attraction ←

Thus Carbon #2 upon displacement from equi is restored & oscillitate about its orig equi position.

So; Think springs, $E_p = \frac{1}{2}kx^2$ & } Model that works well!



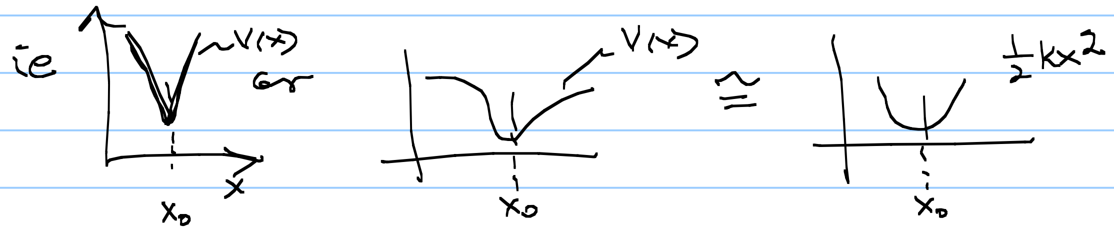
Later on, Townsend Chapt 14, you get a big surprise when you look @

$$\rho_{E\&M} = \frac{1}{\varepsilon_0}\left(E^2 + \frac{1}{\mu_0}B^2\right) = \frac{\text{energy}}{\text{vol}} \quad E\&M$$

that the $\hat{H}_{E\&M}$ also looks like Harmonic Oscillators ie photons!

Also: more complicated potentials can be model to 2nd order as Harmonic Oscillators:

ie



to see this, of course take the Taylor series expan of $V_{general}(x)$ about x_0

$$V_g(x) = \underbrace{V_g(x_0)}_{\text{Energy potential}} + \underbrace{\left.\frac{dV_g}{dx}\right|_{x_0}}_{\text{By def @ min } \frac{dV}{dx}=0} x + \frac{1}{2}\left.\frac{d^2V_g}{dx^2}\right|_{x_0} x^2 + \underset{\text{Higher order terms}}{\text{H.O.T.}}$$

what is really significant is not $|V_g(x)|$ but

$\Delta V_g(x)$ ie potential differences!

So FREE to call this $E_{potential} = 0$

ex: Earth's ground (⏚) = electrical potential ≡ 0 by definition

So

nice!

$$V_g(x) \cong \frac{1}{2}\frac{d^2V_g}{dx^2}x^2$$

Hey $V_g(x) \cong \frac{1}{2}kx^2$

So ... solve Harmonic Oscillator Potential in Q.M.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi = E\psi$$

(well put $\hbar$'s in @ end)

I am going to follow Griffith "Intro to Q.M."
Chpt 2, pg 32

For ~~entire~~ Harmonic Oscillator!

So will use his variables

recall.....

$\Sigma F = m\ddot{x}$

$-kx = m\ddot{x}$

$\ddot{x} + \omega_0^2 x = 0$

$\llcorner\, \omega_0 = \sqrt{\frac{k}{m}}$

so $k = m\omega_0^2$

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi}$$

want to solve this!

2-ways

1) Clever: Algebraic Method
called Ladder $\hat{O}$'s

2.) Brute Force
Solve diffy-Q

of course
results will
be the same!

I.) Algebraic soln to Harmonic Osc time indep Schrö

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

Also... this is how Q.F.T. are done in Fock, #, space

Vacuum ≡ $|0\rangle$

rewrite (just cleaner, Dirac did same thing with E&M field)

$$\frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2\right]\psi = E\psi$$

now these are more $\hat{O}$'s

& will have to be a bit careful

idea: say have $u \pm iv$
then

$$(u-iv)(u+iv) = u^2 + i\underline{vu} - i\underline{uv} + v^2$$

notice I was careful with the order here.

OK so keeping in "mind" order is important, have for v & u = #'s
then

$$(u-iv)(u+iv) = \underbrace{u^2+v^2}_{\text{Looks an awful lot like}}$$

Thus... if we can find these we MIGHT be able to rewrite Schrö's as $(u-iv)(u+iv)=0$
& solve it for each of the multiplicative factors.

OK: Try this ... It:wT (yeah it works!)

$$a_\pm = \frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} \pm im\omega x\right) \Rightarrow$$

ie our
$u \pm iv$

Suggested by look of Schrö

$$\frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2\right]\psi = E\psi$$

now what? remember we assumed

$iuv - ivu = 0$ for u & v = #'s?

here

or u & $v \neq$ #'s but $\hat{O}$'s

so

Test to see if this is

True & we will get a

SURPRISE!

Look. $u = 3$, $v = 2$ $\Rightarrow$ $i(3)(2) - i(2)(3) = 0$

No surprise for ordinary #'s

could rewrite as

$$i[u,v] = i(uv - vu) = 0$$

$$\underbrace{[u,v] \equiv (uv - vu)}$$

called

Commutation relation!

↑ How they commute!

ie You say u & v ordinary # commute! with each other.

But for

u & $v \neq$ # but $\hat{O}$'s

The $[\hat{u}, \hat{v}] \overset{?}{=} 0$

So here we have

$$\hat{a}_{\pm} = \frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} \pm i m\omega x\right)$$

You can ask silly question do $\hat{a}_+$ & $\hat{a}_-$ commute?

i.e.

$$[\hat{a}_-, \hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- \overset{?}{=} 0$$

and it turns out that they do not!

$$[\hat{a}_-, \hat{a}_+] = \hbar\omega$$

So indeed $\hat{a}$'s not same as ordinary #s

But that is not the Big ? or surprise here

at least motivated to look @

$\hat{a}_-\hat{a}_+$ and see what that is .... to see, must try a test function & see what happens--- here is where the surprise comes!

$$\hat{a}_-\hat{a}_+ \underset{\substack{\uparrow \\ \text{test} \\ \text{funct}}}{f(x)} = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{d}{dx} - i m\omega x\right)\left(\frac{\hbar}{i}\frac{d}{dx} + i m\omega x\right) f(x)$$

$$= \frac{1}{2m}\left(\frac{\hbar}{i}\frac{d}{dx} - i m\omega x\right)\underbrace{\left(\frac{\hbar}{i}\frac{df(x)}{dx} + i m\omega x f(x)\right)}_{\substack{\text{note care in} \\ \text{ORDER}}}$$

$$= \frac{1}{2m}\left[-\hbar^2\frac{d^2f}{dx^2} + \underbrace{\hbar m\omega\frac{d}{dx}(xf)}_{\left(\hbar m\omega f + \cancel{\hbar m\omega x\frac{df}{dx}}\right)} - \cancel{\hbar m\omega x\frac{df}{dx}} + (m\omega x)^2 f\right]$$

or

$$(\hat{a}_-\hat{a}_+) f(x) = \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2 + \hbar m\omega\right] f(x)$$

note $f(x)$ is General
never specified
and now can be
removed!

divide from
both
sides

$$\hat{a}_-\hat{a}_+ = \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2\right] + \frac{\hbar\omega}{2}$$

who cares still?

watch

$$\boxed{\left(\hat{a}_-\hat{a}_+ - \frac{\hbar\omega}{2}\right)} = \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2\right]$$

{ That is the
Surprise Because

$\hat{H}\psi = E\psi$ Harm-Osc t

$$\boxed{\frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2\right]}\psi = E\psi$$

OR

$$\left(\hat{a}_-\hat{a}_+ - \frac{\hbar\omega}{2}\right)\psi = E\psi$$ = New

Schrö Equat for ALL Harmonic oscillators!

$$\left(\hat{a}_-\hat{a}_+ - \frac{\hbar\omega}{2}\right)\psi = E\psi$$ = New

Schrö Equat for ALL Harmonic oscillators!

if you're clever, you'll ask
why not have started from

$$\hat{a}_+\hat{a}_- f(x) = ?$$

and

you have gotten to

$$\left(\hat{a}_+\hat{a}_- + \frac{\hbar\omega}{2}\right)\psi = E\psi \Rightarrow$$ Also an equivalent representation of Schrö equat for Harm. Oscil.

$$\underset{\uparrow}{\hat{a}_-\hat{a}_+ - \frac{\hbar\omega}{2}} = \underset{\uparrow}{\hat{a}_+\hat{a}_- + \frac{\hbar\omega}{2}} = \hat{H}_{\text{Harmonic oscillator}}$$

SO

OK, now what?

What have we done?

$$\hat{H}_{H.O.}\psi = E\psi$$

Replaced $$\frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2\right]\psi = E\psi$$

by

$$\hat{H}_{H.O.}\psi = E\psi \Rightarrow \begin{cases} \left[\hat{a}_-\hat{a}_+ - \frac{\hbar\omega}{2}\right]\psi = E\psi \\ \text{or} \\ \left[\hat{a}_+\hat{a}_- + \frac{\hbar\omega}{2}\right]\psi = E\psi \end{cases}$$

OK so how has this algebraic gymnastics helped us?

Well to appreciate answer to that, we must work thru one more exercise.....

Watch this....

$$\hat{H}_{HO}\psi = E\psi$$

Try $\psi \to \hat{a}_+\psi$

$$\hat{H}_{HO}(\hat{a}_+\psi) \overset{?}{=} E_{a_+}(\hat{a}_+\psi)$$

is it still an eigenvalue problem?

↓ use new $\hat{O}$ $\hat{H}_{HO}$

$$\left[\hat{a}_+\hat{a}_- + \frac{\hbar\omega}{2}\right]\hat{a}_+\psi \overset{?}{=} E_{a_+}(\hat{a}_+\psi)$$

$$= \left(a_+a_-a_+ + \frac{\hbar\omega}{2}a_+\right)\psi = E_{a_+}(\hat{a}_+\psi)$$

$$= a_+\left(a_-a_+ + \frac{\hbar\omega}{2}\right)\psi = E_{a_+}(\vec{a}_+\psi)$$

=

$$a_+\left[\left(a_-a_+ - \frac{\hbar\omega}{2}\right)\psi + \hbar\omega\,\psi\right] \overset{?}{=} E_{a+}(\hat{a}_+\psi)$$

wait! recognize this is our our new spissy $\hat{H}\psi$ which $= E_+\psi$

$$a_+\left[E_+\psi + \hbar\omega\,\psi\right] \overset{?}{=} E_{a+}\,\hat{a}_+\psi)$$

$$(E_+ + \hbar\omega)(a_+\psi) = E_{a+}(\hat{a}_+\psi)$$

what have we done?

we got an eigen problem back!

for

$\psi = a_+\psi$

$E_{a+} = E_+ + \hbar\omega$

If we asked $\hat{H}(a_-\psi) \overset{?}{=} E_{a-}(a_-\psi)$

and done the same procedure you'd get

$$(E_+ - \hbar\omega)(\hat{a}_-\psi) = E_{a-}(\hat{a}_-\psi)$$

or for

$\psi = \hat{a}_-\psi$

$E_{a-} = E_+ - \hbar\omega$

So lets recap:

If have Harmonic Oscillator potential:

$$\hat{H}\psi = E_T\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E_T\psi$$

$$\omega^2 = \frac{k}{m}$$

Define $$\hat{a}_\pm = \frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} \pm im\omega x\right)$$

Then $$\hat{H}\psi = E_T\psi$$

$$\left(a_+a_- - \frac{1}{2}\hbar\omega\right)\psi = E_T\psi$$

and $$(E_T+\hbar\omega)(a_+\psi) = E_{a+}(a_+\psi)$$

$$(E_T-\hbar\omega)(a_-\psi) = E_{a-}(a_-\psi)$$

So: if you know ψ_1 & E_1 say

Then $$(E_1+\hbar\omega)(a_+\psi_1) = E_{a+}(a_+\psi_1)$$

→ So apparently $a_+\psi_1$ is raising ψ_1 to next energy state ψ_2

$$(E_1+\hbar\omega)\psi_2 = E_2\psi_2$$

& $$E_2 = E_1+\hbar\omega$$

∴

& could also do

$$(E_2+\hbar\omega)(\hat{a}_+\psi_2) = E_{a+}(\hat{a}_+\psi_2)$$

or

$$(E_2+\hbar\omega)\psi_3 = E_3\psi_3$$

so $\psi_3 = a_+\psi_2$

& $E_3 = E_2+\hbar\omega$

$\hat{a}_+$ = raising $\hat{o}$!

Similarly say start @ ψ_3

$$(E_3-\hbar\omega)(a_-\psi_3) = E_{a-}(a_-\psi_3)$$

$a_-\psi_3$ Lowers state

$$(E_3-\hbar\omega)\psi_2 = E_2\psi_2$$

so $\psi_2 = a_-\psi_3$

& $E_3 = E_3-\hbar\omega$

YEAH!

Thats our TRICK

$\hat{a}_+$ = raising $\hat{o}$; $\hat{a}_+\psi_n = \psi_{n+1} \Rightarrow E_{n+1} = E_n+\hbar\omega$

$\hat{a}_-$ = lowering $\hat{o}$; $\hat{a}_-\psi_n = \psi_{n-1} \Rightarrow E_{n-1} = E_n-\hbar\omega$

How could we even do anything useful?

Remember, we are trying to solve Harmonic Oscill Problem --- if there were an easier way, we'd do it. Also, this raising & lowering $\hat{o}$ formalism IS how Q.F.T. is done in a Fock # space!

$$\underbrace{a_+ a_- |0\rangle}_{\text{vacuum}} = \underbrace{a_+ |-1\rangle}_{\text{antiparticle}} = \underset{\substack{\uparrow \\ \text{particle} \\ + \\ \text{antiparticle} \\ \text{annihilation}}}{|0\rangle}$$

Enough, lets do one of our problems w/ this Technique.

To do so, need to realize 1 more thing.

Say you have ψ_n

then lower it $a_- \psi = \psi_{n-1}$

then again $a_- \psi_{n-1} = \psi_{n-2}$

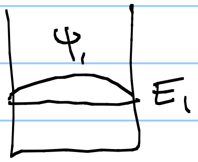
&

again

⋮

How long can you do this?

Clearly, at least in Schrö non relativistic equation, there is a bottom



one less; $a_- \psi_1 = \psi_0 = 0$

* Q.F.T. based on Dirac equat & neg energies must reconcile this argument by defining (?) maybe the vacuum $= |0\rangle$ & then (−) energies are allowe

So, at some point when Lowering states

$$\boxed{a_-}\psi_0 = 0$$

call this lowest Harm Osc state, not the ψ_1 w/ square well

Can we do something with that?

$$\boxed{\frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} - im\omega x\right)}\psi_0 = 0$$

$$\frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d\psi_0}{dx} - im\omega x\right) = 0$$

$$\frac{d\psi_0}{dx} = \frac{-m\omega}{\hbar}x\psi_0$$

$$\int\left[\frac{d\psi_0}{\psi_0} = \frac{-m\omega}{\hbar}x\,dx\right]$$

$$\int\frac{d\psi_0}{\psi_0} = \int\left(\frac{-m\omega}{\hbar}\right)x\,dx$$

$$\ln\psi_0 = \frac{-m\omega}{2\hbar}x^2 + C$$

$$\psi_0(x) = e^{c}\,e^{\frac{-m\omega}{2\hbar}x^2}$$

$$\boxed{\psi_0(x) = A_0 e^{\frac{-m\omega}{2\hbar}x^2}}$$

Cool!

we just found ground state of Harm. Oscillator.

$\psi_0 = A_0 e^{-\frac{m\omega}{2\hbar}x^2}$ = graund state H.O.

what is it's energy?

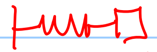
well $\hat{H}\psi_0 = E_0\psi_0$

$$\underbrace{\left(a_+a_- + \frac{\hbar\omega}{2}\right)}_{\hat{H}}\psi_0 = E_0\psi_0$$

$$a_+(\underbrace{a_-\psi_0}_{0}) + \frac{\hbar\omega}{2}\psi_0 = E_0\psi_0$$

so $E_0 = \frac{\hbar\omega}{2}$ = ground energy of H. Osc.

not grand, Lowest energy is Not Zero!

Classically  can be still But not true in Q.M.

idea: is related to uncertainty principle!

Conclude: All Harmonic Oscillators have energy allways, never Zero. The vacuum of H.Osc in ground state is loaded w/ energy.

> Back to us, we ψ_0 & now E_0 so can get all ψ_n & E_n by using our raising $\hat{O}$

$$\psi_n(x) = A_n(a_+)^n e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\& \quad E_n = \left(n+\frac{1}{2}\right)\frac{\hbar}{\omega}$$

Wow!

Most Elegant!

Note: ψ_n only to w/in Normaliz constant.

EXAMPLE:

what are ψ_1 & E_1 of a H. Osc?

$$\psi_1 = A_1 a_+ \psi_0$$

$$= A_1 \underbrace{\frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} + imωx\right)}_{a_+} \underbrace{e^{-\frac{mω}{2\hbar}x^2}}_{\psi_0}$$

$$= \frac{A_1}{\sqrt{2m}}\left[\frac{\hbar}{i}\left(\frac{-mω}{\hbar} x\, e^{\frac{-mω}{2\hbar}x^2} + imωx\, e^{\frac{-mω}{2\hbar}x^2}\right)\right]$$

$$\psi_1(x) = \left(iA_1 ω\sqrt{2m}\right) x\, e^{-\frac{mω}{2\hbar}x^2}$$

recall: observables $= \langle \hat{o} \rangle$ $\propto \psi^* \psi$ so i's cancel

& $E_1 = \left(1+\frac{1}{2}\right)\hbar ω = \frac{3}{2}\hbar ω$

Most Elegant....

gen $\psi_{100}(x)$ & E_{100} is you like!

OK: That was elegant, sophisticated
& you will use it again & again

E& Field Quantization

Q.F.T

But Could have just solved

$$\hat{H}\psi = E_n\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E_T\psi$$

The Brute Force way!

Not Fun.... See Scherrer, see Griffiths.
Many Tricks
Not so elegant
doesn't lend itself in general fashion:

must define $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$

rewrite Schö in term of derivs of of $\frac{d}{d\xi}$

$$\frac{d^2\psi}{d\xi^2} = (\xi^2 - K)\psi$$ do some magic & get

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2}$$

where $H_n(x) \equiv$ Hermite Polynomials

* this is normalized
That's good

$H_0(x) = 1$
$H_1(x) = 2x$
$H_2(x) = 4x^2 - 2$
$H_3(x) = 8y^3 - 12y$

and so on.

How did

raising-lower Technique = Brute Force?

$$\psi_0 = A_0 \boxed{e^{-\frac{m\omega}{2\hbar}x^2}} = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_0(\xi)\, e^{-\xi^2/2}$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2}} (1) \boxed{e^{\frac{m\omega}{2\hbar}x^2}}$$

yup

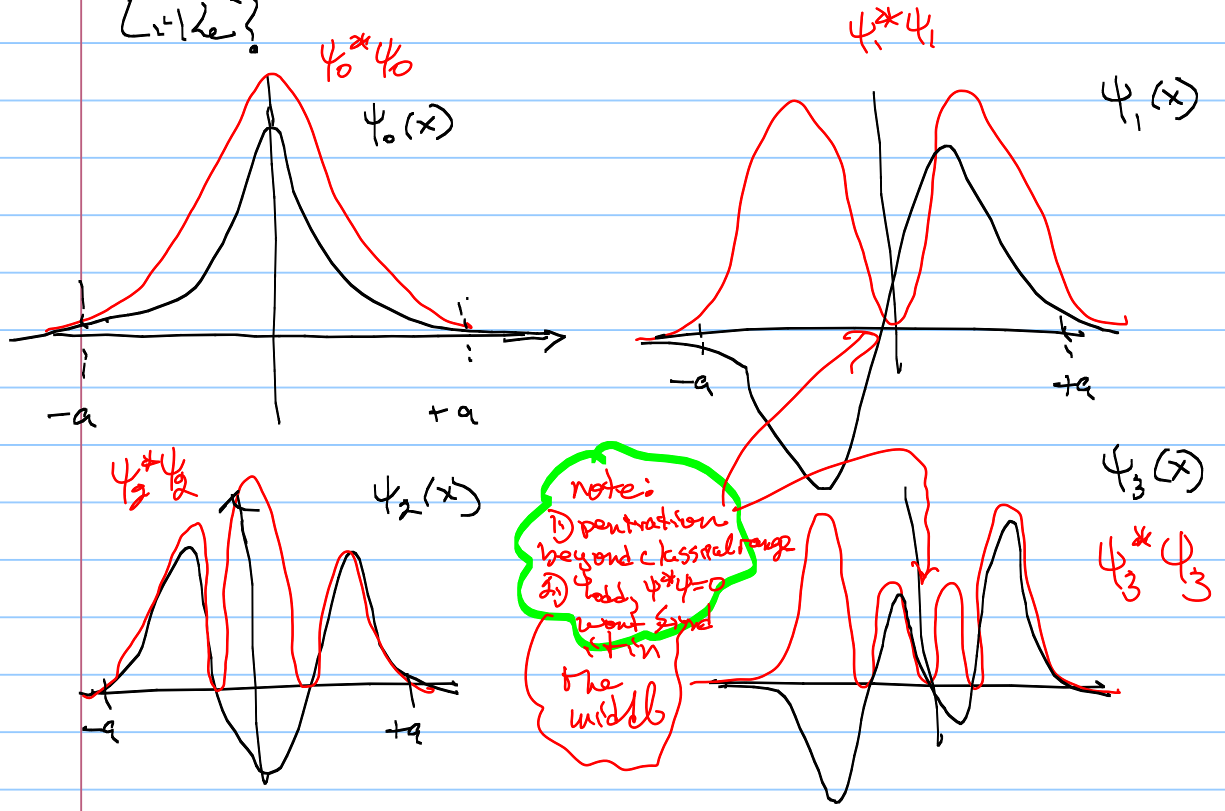
O-K Nice part of this form is that it is Normalized!

So, what do $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E_T \psi$

$\omega = \sqrt{\frac{k}{m}}$

$\psi_n(x) = A_n (a_+)^n e^{-\frac{m\omega}{2\hbar}x^2}$; $E_n = (1+\frac{1}{2})\hbar\omega$

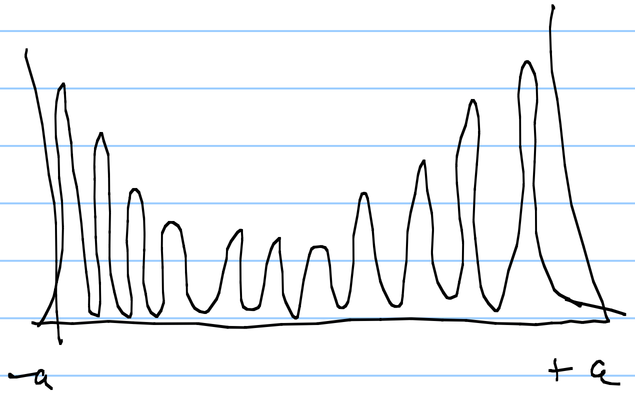
solutions look like?



note:
1) pentration beyond classical range
2) ψ_{odd}, $\psi^*\psi=0$ wont find it in the middle

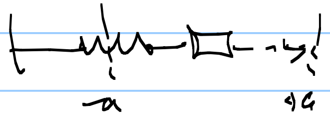
Then

$\psi_{100}^*\psi_{100}$



* Note $n\to\infty$ looks a bit like classical

w/ prob of finding particle @

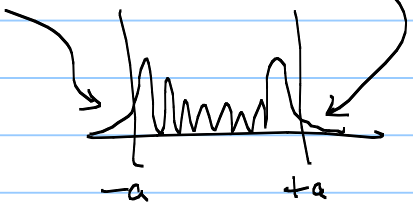


limits where vel = 0

max!

Griffith's notes:

1.) Q.M Harmonic oscillitor also can penetrate <u>beyond</u> its classical limits (ie range for its energy)



2.) Weird for
$n = \text{odd}$
$\psi_n^* \psi_n = 0$

won't ever find Q.M. H. Osc in the center!

3.) only when $n = \text{Huge}$ do you start seeing classical resemblance!

E.F. Downey: H. Osc't from Griffiths Intro to Q.M.

E.T. Doheny / BSC Physics: PH402 Quantum

Harmonic oscillator recap

Note Title 4/12/2006

Townsend!

So is you have

$$\hat{H}\psi_n = E_n \psi_n$$

$$\left(\frac{p^2}{2m} + \frac{1}{2}kx^2\right)\psi_n = E_n \psi_n$$

= Harmonic Oscillator Potential

ψ_n & Hermite Polynomials

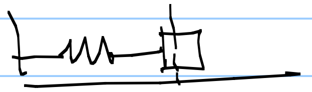
and you get energies

$$E_n = (n+\tfrac{1}{2})\hbar\omega$$

This is done Great quantitatively in Townsend pg 205

1. * Energy can't = 0! (ie it would violate uncertainty relation)

$$E_{Lowest} = (0+\tfrac{1}{2})\hbar\omega = \frac{\hbar\omega}{2}$$



classically Lowest energy is x=0

→ If x = 0 exactly you'd think this is min in energy because the string is unstretched

But is x=0 exactly, $\Delta x = 0$

Then δp must be $\frac{\hbar/2}{\Delta x}$ not zero.

So by the uncertainty princ, you cannot have Zero --- min energy of Harm Oscill.

2) →

2.) also $E_{H.O.} = E_n = (n+\frac{1}{2})\hbar\omega$

note your energy comes in
discrete jumps $= \hbar\omega$

$\frac{1}{2}\hbar\omega$, $n=0$; ψ_0

$1\frac{1}{2}\hbar\omega$, $n=1$; $\psi_1 \Rightarrow 1\hbar\omega$

$2\frac{1}{2}\hbar\omega$; $n=2$; $\psi_2 \Rightarrow 2\hbar\omega$

$\vdots$

$n\frac{1}{2}\hbar\omega$; ; $\psi_n \Rightarrow 3\hbar\omega$

↑

These
quantized
energies

Characteristic of all Harmonic Oscillators:

ex: Townsend Chapt 14

$$\hat{H}_{E\&M} = \frac{1}{8\pi}\int d^3\vec{r}\left[\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right]$$

$$= (\text{DIRAC's equations})$$

$$\left.= \sum_{k,\lambda} \hbar\omega\left(\hat{a}^{\dagger}_{k,\lambda}\hat{a}_{k,\lambda} + \frac{1}{2}\right)\right\} = \text{exactly the same as}$$

↪ the $\hat{H}_{\text{Harmonic oscillator}}$ using ladder $\hat{O}$'s!

Therefore; E&M = Harmonic oscillators characterized by energies

$$(n+\tfrac{1}{2})\hbar\omega$$

$n=0$ $\quad E = \hbar\omega/2$ } no Zero, Zero point energy for E&M Field!

$n=1$ $\quad (1+\tfrac{1}{2})\hbar\omega$

$n=2$ $\quad (2+\tfrac{1}{2})\hbar\omega$

$\vdots$

$(n+\tfrac{1}{2})\hbar\omega$

n = # of lumps of energy = photons!

The vacuum is not Boring!

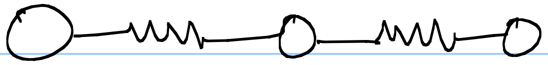
$|0\rangle$ = some energy

in fact

it has $\frac{\hbar\omega}{2}$ for each k running to ∞

$\therefore$ Vacuum = ∞ energy in its ground state!

Similarly



Solids clearly

$$\hat{H}_{\text{solid}} = \hat{H}_{\substack{\text{Harmonic} \\ \text{oscill}}}$$

Thus $E_n = (n + \frac{1}{2})\hbar\omega$

$\uparrow$

These discrete energy jumps

$= \underline{\underline{\text{Phonons}}}$