

Derivation of time indep Schröd

Start w/ time indep Schröd

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\vec{r}, t) \Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

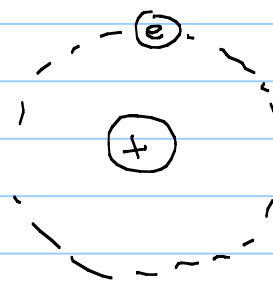
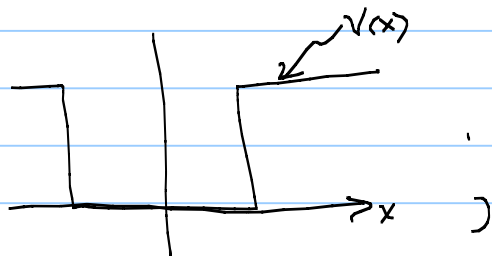
Generally, can see that it is $V(\vec{r}, t)$
that is the trouble

↳ i.e. that

distinguishes the different
problems

$-\frac{\hbar^2}{2m} \nabla^2 \Psi$ doesn't
change
problem to
problem

So: lets consider $V(\vec{r}, t)$, the potential
energy.



$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

Notice V in these Blc cases $\neq f(t)$

So, we will restrict ourselves to problems
 $V \neq f(t)$

focus not to be as limiting ($V(\vec{r}, t) \rightarrow V(\vec{r})$)
as you think.
You can add time dependence later.

Our goal for now is to find legitimate
but more simple general cases & soln's

$\Psi(x, t)$'s that form a Complete
Basis SET
↑
note
sometimes
 \vec{r}, t or x, t
↓
? why

Cause once we get a good basis, we
can use it for more difficult
problems to build from

- 1) perturbation theory
- 2) $\frac{1}{2}$ time dependent perturbation theory.

So lets solve $V \neq S(t)$

$\frac{1}{2}$ then one more thing $\dots \rightarrow$

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi(x,t) + V(x) \Psi(x,t) = i\hbar \frac{\partial \Psi(x,t)}{\partial t}$$

↑
1st

Next... We are only going to look for

$\Psi(\vec{r}, t)$ that are eigenfunctions of the total energy operator

$$\text{i.e. } i\hbar \frac{\partial \Psi(x,t)}{\partial t} = E_{\text{TOT}} \Psi(x,t) = E_{\text{energy}} \text{ eigenvalue.}$$

This \Rightarrow 's $\Psi(x,t)$ eigenfunction will be "States" of definite Energy E_{TOT} .

\Rightarrow Here goes:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = E_{\text{TOT}} \Psi(x,t)$$

lets guess @ solu

$$\Psi(x,t) = \Psi(x) g(t)$$

Note, from guess this

$$\Psi \rightarrow \Psi(x,t)$$

$$\Psi \rightarrow \Psi(x)$$

This solu assumes

that Ψ can be built from

two separable solns

$$\Psi \rightarrow \Psi(x) \text{ only } \&$$

$$g \rightarrow g(t) \text{ only}$$

and that $\Psi =$ the product of these.

Why Not? If it did work, oh well.

But more important \Rightarrow

indeed it is possible to show that separable solns form a complete basis set of solns - ... just what we were looking for.

$$\text{so } \Psi = \psi g \quad i\hbar \frac{\partial \Psi}{\partial t} = E_T \Psi$$

$$i\hbar \frac{\partial (\psi g)}{\partial t} = E_T \psi g$$

$$\frac{1}{\psi} \left[\psi \left(i\hbar \frac{\partial g}{\partial t} \right) = \psi (E_T g) \right]$$

Trick
watch
what
happens

$$\frac{1}{g} i\hbar \frac{dg}{dt} = E_T$$

we've
separated
 ψ out!

to solve this diff-eq
separate variables

note: $\Rightarrow \frac{dE}{g} = \frac{1}{i\hbar} E_T dt$
 ∂ partial \rightarrow
 full derivs

$$\int [\quad] \text{ indef integral}$$

$$\ln g = -\frac{i}{\hbar} E_T t$$

or

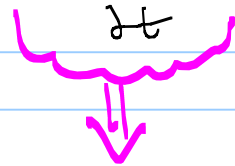
$$g = e^{-\frac{i E_T t}{\hbar}}$$

$$g(t) = e^{-\frac{i E_T t}{\hbar}}$$

just solved
time dep part
of Ψ = energy
eigenstate

Now back to full problem

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} (x,t) + V(x) \Psi(x,t) = i\hbar \frac{\partial \Psi(x,t)}{\partial t}$$



But wait

we already said looking for energy eigenstates where

$$i\hbar \frac{\partial \Psi}{\partial t} = E_{\text{tot}} \Psi$$

so

that means

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi = E_{\text{tot}} \Psi$$

Note $\Psi = \Psi(x,t)$

use my sep variables soln

$$\Psi = \psi(x) g(t)$$

substitution

$$\psi g \left[-\frac{\hbar^2}{2m} \frac{\partial^2 (\psi g)}{\partial x^2} + V \psi g = E_{\text{tot}} \psi g \right]$$

once again ... trick

$$\frac{1}{\psi g} \left[g \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V g \psi = E_{\text{tot}} \psi g \right]$$

$$\frac{1}{\psi} \left(\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \right) + V = E_T$$

or

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x) \psi(x) = E_T \psi(x)$$

This is time indep Schrö

Note subtle differences

Time
dep
Schrö

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x) \Psi(x,t) = E_T \Psi(x,t)$$

your goal is to solve

$\Psi(x,t)$ all @ once

→ Now your goal is to just
solve for $\psi(x)$ = only $\psi(x)$ part

So instead of solving time dep Schröd
 $\Psi(x,t)$

Solve time indep Schröd for $\psi(x)$
then

to complete your Total Wave Function,
Simply multiply it
by your time dep
part
which is always
the same!

So, Real

PROBLEMS: $\Rightarrow \hat{H} \Psi(x,t) = i\hbar \frac{\partial \Psi(x,t)}{\partial t} \Leftarrow$ time dep Schröd
Look like try

$$\hat{H} \psi(x) = E_{\text{tot}} \psi(x)$$

time indep Schröd

solve for $\psi(x)$

easier!

then $\Psi(x,t) = \psi(x) e^{-iE_{\text{tot}}t/\hbar}$

The same for all

$V(x,t) \rightarrow V(x)$
separable,
energy eigenvalue
solutions!

So in general will spend lots of time solving time indep Schrod - energy eigen value problem (ie states of definite energy)

$$\hat{H}\psi = E_n \psi$$

As mentioned the ultimate $\Psi = \psi e^{-i\frac{E_n}{\hbar}t}$

which

are perfect for building more complicated problems.

These Ψ 's are called "stationary" $\Rightarrow \psi \neq \Psi(t)$

but clearly there is $\Psi(t)$.

Idea of stationary is in the expectation values...

watch

$$\langle O \rangle = \int \Psi^* \hat{O} \Psi dx$$

$$= \int \psi(x)^* e^{+i\frac{E_n}{\hbar}t} \hat{O} \psi(x) e^{-i\frac{E_n}{\hbar}t} dx$$

Never changes

$$e^{+i(\cdot)} e^{-i(\cdot)} = 1$$

$$\langle O \rangle = \int \psi(x)^* \hat{O} \psi(x) dx$$

\hookrightarrow ie the expectation values of OBSERVABLES DO NOT CHANGE in time... they are stationary!

Ψ of course does depend on t , but all
The expectations do not!

Is this a problem, cause in real life
measurable things do change in time?

Answer: No:

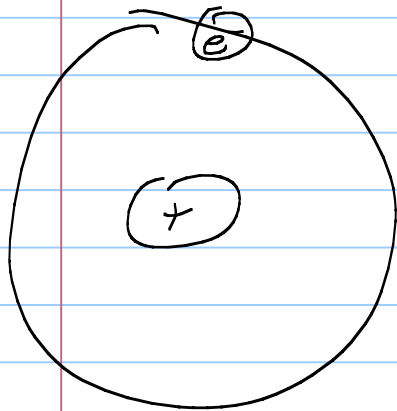
Again these $\Psi = \psi e^{-iE_t/\hbar}$
= Energy eigenstates
= energy eigenvalue
solns

Are a good basis to build other
complete solns.

So later: will use these Ψ 's to build
solns to tougher problems!

Ex:

H-atom



$$\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$



$$\hat{H}\Psi = E_{\text{Tot}}\Psi$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{4\pi\epsilon_0} \frac{1}{r} (-e) = E_{\text{Tot}}\Psi$$

Solve for $\Psi(r)$

$$\Psi_{\text{H-atom}} = \Psi(r) e^{-i \frac{E_{\text{Tot}}}{\hbar} t}$$

↑
all of
chapt 6

* Separation of Variables II

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V \Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

assume $\Psi = \psi(x)g(t)$

$$\left[\psi \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V \psi g = \psi i\hbar \frac{\partial g}{\partial t} \right] \frac{1}{\psi g}$$

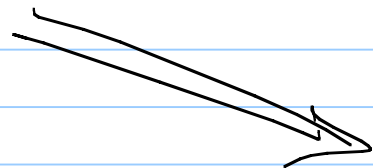
$$\underbrace{\frac{1}{\psi} \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)}_{\text{S}(x) \text{ only}} = \underbrace{\frac{1}{g} i\hbar \frac{\partial g}{\partial t}}_{\text{S}(t) \text{ only}}$$

Wo-ah ... we've separated variables!

S(x) only

S(t) only

Now if you think about this for a while, you get cool conclusion



if you have

$$f(x) = f(t)$$

x & t are truly independent variables

then try this

say

$$\underbrace{f(x=2)}_{\text{OK}} = f(t)$$

↑
|

try then

$$f(t=2)$$

But why not

$$f(t=3)$$

$$\text{or } f(t=4)$$

get the idea that the only way you could have this equality

$$f(x) = f(t)$$

for all x & t
would be
if & only if

$$f(x) = f(t) = a \text{ (constant)}$$

for all of x & t

So that constant we could recognize from \rightarrow

$$\frac{1}{\psi} \left(-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x) \right) = \frac{1}{g} i\hbar \frac{dg}{dt} = C$$

So
$$\frac{1}{\psi} \left(-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x) \right) = C$$

1.)
$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = C\psi$$

and

2.)
$$\frac{1}{g} i\hbar \frac{dg}{dt} = C$$

$$\int \frac{i\hbar dg}{g} = \int C dt$$

$$\ln g = \frac{-i}{\hbar} C t$$

$$g(t) = e^{-i \frac{C}{\hbar} t}$$

Now C... what could C be?

note units must be Energy!

So establish $C = E_T$

Then

1.)
$$g(t) = e^{-i \frac{E_T}{\hbar} t}$$

2.)
$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = E_T \psi \Leftarrow \text{Hey time indep Schröd}$$

Cool!