

E.F.D. Demery, BSC Physics: $\hat{S}_x, \hat{S}_y \Rightarrow$ eigenvectors & values in $|\pm z\rangle$

Note Title

2/17/2007

All right we've got

$$\hat{S}_z = \frac{\hbar}{2} \sigma_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_x = \frac{\hbar}{2} \sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{\hbar}{2} \sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

all in
 $|\pm z\rangle$
Basis!

we have $S_z |\pm z\rangle = \pm \frac{\hbar}{2} |\pm z\rangle$

eigenvectors: $|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\uparrow\rangle$

$| -z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle$

w/ eigenvalues $+\frac{\hbar}{2}$ & $-\frac{\hbar}{2}$ respect.

? now is what are eigenvectors & values for \hat{S}_x & \hat{S}_y

HERE'S how! = Solving any old eigen value problem!

\hat{S}_x : $\hat{S}_x |\psi\rangle = c |\psi\rangle = \text{eigen}$

matrix rep in $|\pm z\rangle$ basis

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = c \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

has to be 2 component
for matrix mult to work.

OK $\hat{S}_x |\psi\rangle = c |\psi\rangle$



$$\begin{pmatrix} S_x - c \end{pmatrix} |\psi_x\rangle = 0$$

$$\begin{pmatrix} S_x - \mathbb{1} c \end{pmatrix} |\psi_x\rangle = 0 \quad \left. \vphantom{\begin{pmatrix} S_x - \mathbb{1} c \end{pmatrix} |\psi_x\rangle = 0} \right\} \text{OK how to solve this...}$$

How?

idea from Townsend

$ab = 0$
 $a \text{ or } b = 0$
 But is for all $a \neq 0$
 $ab \text{ still } = 0$
 Then indeed $b = 0$

OK: rewrite as

$$\underline{M} |\psi_x\rangle = 0$$

1) if $\det \underline{M} \neq 0$, then \underline{M}^{-1} exists

so then $\underline{M}^{-1} (\underline{M} |\psi_x\rangle = 0)$

$$\underbrace{\underline{M}^{-1} \underline{M}}_{\mathbb{1}} |\psi_x\rangle = 0$$

$|\psi_x\rangle = 0 = \text{trivial soln, no help.}$

2) if $\det \underline{M} = 0$, then No \underline{M}^{-1}

if no \underline{M}^{-1} exists then

$$\underline{R} (\underline{M} |\psi_x\rangle = 0)$$

All possible matrices
 not $= 0$

\Rightarrow
 $ab = 0$
 \uparrow For all a
 $\therefore b = 0$

so $\underline{M} |\psi_x\rangle = 0$
 will be non trivial soln
 for $\det \underline{M} = 0$

need to think about that but key to solving eigen value problems!

$$\text{So } \sum_x M |\psi_x\rangle = 0$$

has solns iff

$$\det |M| = 0$$

or we really had

$$\left(\sum_x -1 C \right) |\psi_x\rangle = 0$$

so

so/ve

$$\det \left(\sum_x -1 C \right) = 0$$

$$\det \left| \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \right| = 0$$

$$\det \begin{vmatrix} -C & \frac{1}{2} \\ \frac{1}{2} & -C \end{vmatrix} = 0$$

OK

$$C^2 - \left(\frac{1}{2}\right)^2 = 0$$

$$C^2 = \frac{1}{4} \quad \therefore C_1 = \sqrt{\frac{1}{4}}$$

$$C_1 = +\frac{1}{2}, \quad C_2 = -\frac{1}{2} = \sum_x \text{ eigen-values}$$

We've got \sum_x eigenvalues $\pm \hbar/2$, now lets get the eigenvectors!

here's how: $S_x |\psi_x\rangle = c |\psi_x\rangle$

Solve this for each eigenvalue

$$c_1 = +\hbar/2$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Clearly not dependent: But to be true $\psi_1 = \psi_2$

$$\text{so } |\psi_x\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \text{ try } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{or } |\psi_x\rangle \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

* Recall

Normalization!

$$\langle \psi_x | \psi_x \rangle = 1$$

$$\text{so } (1 \ 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \dots \text{oops}$$

$$|\psi_x\rangle = N \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{Then } \langle \psi_x | \psi_x \rangle = (N \ N) \begin{pmatrix} N \\ N \end{pmatrix} = 2N^2 = 1 \quad \text{OR } N = \frac{1}{\sqrt{2}}$$

that's it

$$|\psi_x\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{See } +\frac{\hbar}{2}$$

See $-\frac{\hbar}{2}$, $|\psi_x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

So

$$|+x\rangle_{sz \text{ basis}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{w/ eigen val} = +\frac{\hbar}{2}$$

$$|+x\rangle = \frac{1}{\sqrt{2}} (|z\rangle + |-z\rangle)$$

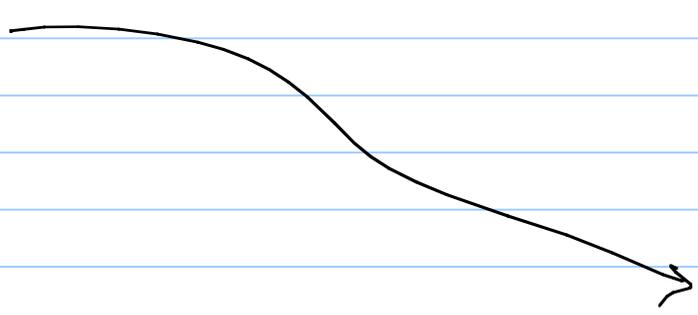
$$|-x\rangle_{sz \text{ basis}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{w/ " " } = -\frac{\hbar}{2}$$

$$|-x\rangle = \frac{1}{\sqrt{2}} (|z\rangle - |-z\rangle)$$

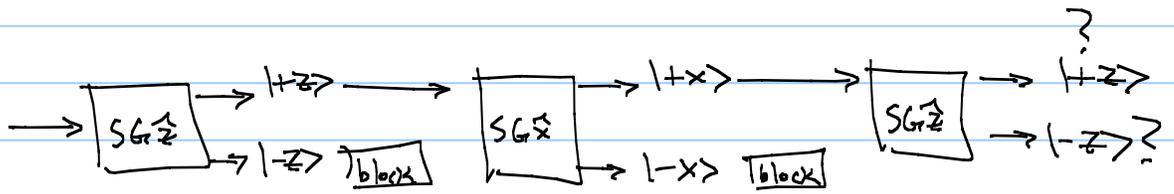
recall $|+z\rangle_{sz \text{ basis}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ w/ " " $\frac{\hbar}{2}$

$$|-z\rangle_{sz \text{ basis}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{w/ " " } = -\frac{\hbar}{2}$$

OK... Recall



our SG exper



Solution!

$$|+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so

proj of $|+z\rangle$ on initial state $|+x\rangle$?

well

Measurement DESTROY'S the state & make something new!

For this S.G. experiment BUT TRANSPARENT FOR FULL DETECT

See Solver 173-176

$$\langle +z | x \rangle$$

$$(1 \ 0) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}$$

or

$$|\langle -z | x \rangle|^2 = \left| (0 \ 1) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 = 50\%$$

Then prob of finding $|+x\rangle$ in $|+z\rangle$?

$$|\langle z | x \rangle|^2 = \frac{1}{2} = \boxed{50\%}$$

That's Q.M. that's weird Entirely non-classic! & 100% proven experimentally!

Wow! Started out w/ only $|+z\rangle$

& went thru a SG_x (just a measurement) & now do SG_z again

But get 50% now = $|+z\rangle$!

Homework!

Onward ... Find eigenvectors & eigen values of \hat{S}_y in $|\pm z\rangle$ basis!

OK: $\hat{S}_y |\psi\rangle = c |\psi\rangle$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = c \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

or

$$(\hat{S}_y - 1c) |\psi\rangle = 0$$

$$\begin{pmatrix} -c & -i\hbar/2 \\ i\hbar/2 & -c \end{pmatrix} |\psi\rangle = 0$$

only solns for $\det \begin{vmatrix} -c & -i\hbar/2 \\ i\hbar/2 & -c \end{vmatrix} = 0$

or $c^2 - \left(\frac{\hbar^2}{4}\right) = 0$

$$c^2 = \frac{\hbar^2}{4}$$

$$c_1 = \hbar/2, \quad c_2 = -\hbar/2$$

For $\hbar/2$: $\hat{S}_y |\psi\rangle = \frac{\hbar}{2} |\psi\rangle$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

or $\begin{pmatrix} -i\psi_2 \\ i\psi_1 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

... huh what is?

$$\psi_1 = \psi_2$$

$$-i\psi_1 = \psi_2$$

$$i\psi_1 = \psi_2$$

no good

what is $\psi_1 = i\psi_2$

$$-i\psi_2 = i\psi_2$$

$$-\psi_2 = \psi_2$$

nope

$$i(i\psi_2) = \psi_2$$

$$-\psi_2 = \psi_2$$

Try $\Rightarrow \psi_1 = -i\psi_2$

$$-i\psi_2 = -i\psi_2$$

$$\psi_2 = \psi_2$$

$$i(-i\psi_2) = \psi_2$$

$$\psi_2 = \psi_2$$

OK!

$$\sum_{j=0, \dots, n} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

is

$$\psi_1 = -i\psi_2 \quad \text{or} \quad \psi_2 = \frac{1}{i}\psi_1 = i\psi_1$$

$$\text{So } |\psi\rangle = \begin{pmatrix} \psi_1 \\ i\psi_1 \end{pmatrix} \text{ say } = N \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\text{Then } \langle \psi_2 | \psi_2 \rangle = 1$$

$$(N \quad -iN) \begin{pmatrix} N \\ iN \end{pmatrix} = N^2 - i^2 N^2 = 1$$

$$2N^2 = 1 \quad N = \frac{1}{\sqrt{2}}$$

$$\therefore |\psi\rangle \text{ has eigen value } +\frac{\hbar\omega}{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\text{For } -\frac{\hbar\omega}{2}; |\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$\sum_0 \dots \dots \dots$ in $|\pm z\rangle$ basis

eigen
value
↓

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1|+z\rangle + 0|-z\rangle = |\uparrow\rangle; \hbar/2$$

$$|-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0|+z\rangle + 1|-z\rangle = |\downarrow\rangle; \hbar/2$$

$$|+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (|+z\rangle + |-z\rangle) = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle); \hbar/2$$

$$|-x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (|+z\rangle - |-z\rangle) = \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle); \hbar/2$$

$$|y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} (|+z\rangle + i|-z\rangle) = \frac{1}{\sqrt{2}} (|\uparrow\rangle + i|\downarrow\rangle); \hbar/2$$

$$|-y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}} (|+z\rangle - i|-z\rangle) = \frac{1}{\sqrt{2}} (|\uparrow\rangle - i|\downarrow\rangle); \hbar/2$$

NOTE: That $|\pm z\rangle$ are linearly indep of each other

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{as } |\pm x\rangle \rightarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\& \quad |\pm y\rangle \rightarrow \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

*

in 3-D
we need
complex
#s to
get
SPIN RIGHT!

So, we've managed to build 3 representations
of 2 linearly indep basis.

to do so... we were required to go into the
complex plane! Entirely non classic.