

OK, we've got

$$\hat{S}_z \nparallel |\pm z\rangle \text{ in an } S_z \text{ basis} \quad \begin{array}{l} \text{(built from)} \\ \text{results of} \\ S.G.\hat{z} \end{array}$$

Started w/ CAVS of spin $\frac{1}{2}$ system

$$|\pm z\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\nparallel |\downarrow\rangle = |\downarrow\rangle = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{eigenectors}} \quad \begin{array}{l} \text{= complete} \\ \text{(matrix rep)} \end{array}$$

Basis spanning
Spin Space

\nparallel got

$$\hat{S}_z = \begin{pmatrix} \langle z | S_z | z \rangle & \langle z | S_z | -z \rangle \\ \langle -z | S_z | z \rangle & \langle -z | S_z | -z \rangle \end{pmatrix} = \begin{pmatrix} \chi_{1z} & 0 \\ 0 & \chi_{1z} \end{pmatrix}$$

\uparrow
matrix
rep

recovering SG \hat{z} results

$$\hat{S}_z |\pm z\rangle = \pm \chi_{1z} |\pm z\rangle$$

KEEP IN MIND: SPIN = ANGULAR MOMENTUM!

↳ knowing from

$$\text{Q.M.}, \vec{L} = \vec{r} \times \vec{p} \rightarrow (\vec{x})(i\hbar \frac{\partial}{\partial x})$$

that $\left[L_x^2, L_z \right] = 0$

$\left\{ \begin{array}{l} \left[L_x, L_z \right] \neq 0 \\ \left[L_x^2, L_z \right] = 0 \end{array} \right\}$

⇒ is eigenfunctions
vectors
of $L_x^2 \& L_z$

= max. the formative

Q.M. state of

angular momentum!

NICE!

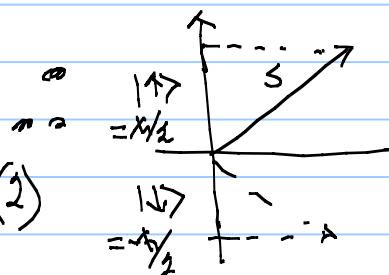
SPIN = Intrinsic angular momentum?

not from fund. sources of $\vec{r} \times \vec{p}$

But instead Spin ang momentum is itself a fundamental source of ang momentum, \hbar !

so $[S^2, S_z] = 0$

$\left[S_x, S_z \right] \neq 0$



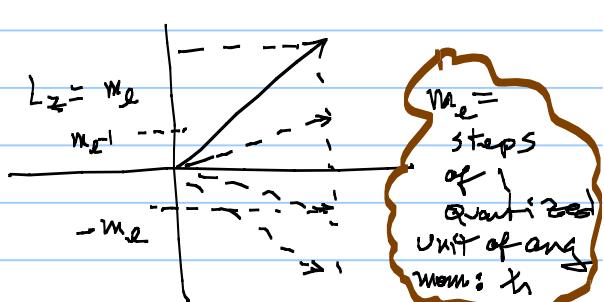
only (2)

$S^2 |S m_s\rangle = s(s+1)\hbar^2 |S m_s\rangle$

$S_z |S m_s\rangle = m_s \hbar |S m_s\rangle$

or just

$S_z |\pm z\rangle = \pm \frac{\hbar}{2} |\pm z\rangle$



$$L^2 |S m_e\rangle = l(l+1)\hbar^2 |S m_e\rangle$$

$$L_z |S m_e\rangle = m_e \hbar |S m_e\rangle$$

So here $\hat{S}_z \hat{\psi} |+\rangle = |\pm\frac{\hbar}{2}\rangle$

\uparrow operator \uparrow eigenvectors \uparrow eigenvalues
 ↑ normalized.

Note $|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is it normalized?

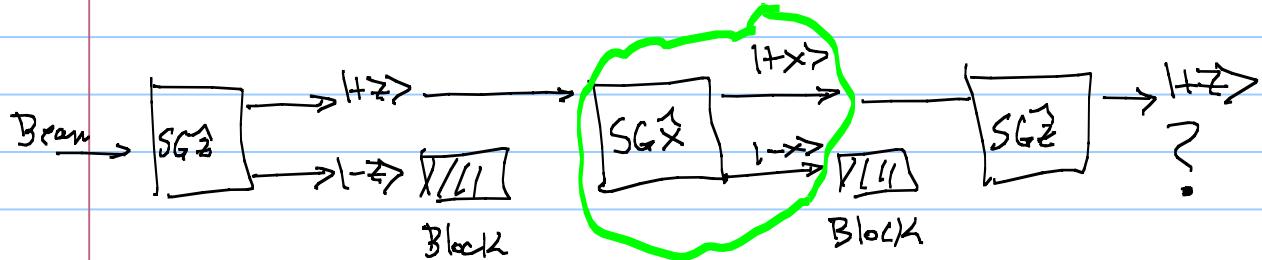
$\langle z|z\rangle = (1^*)^*(1) = 1$ yup.

Now the? \hat{S}_x, \hat{S}_y 's

look like what are the eigenvalues & eigenvectors of these

ALL in a S_z eigen basis!

Why? Investigate QM



Note: Space Quantization, as soon as particles hit SG_x , they must quantize into $|+\rangle$ (ie 1 or in x dir) states.

So we'd like to

know, is $\langle z|x\rangle = 1$ (ie prob = 1 because we started w/ $|+\rangle$?)

So goal is, w/ $\hat{S}_z \notin \{\pm\}$, get $\hat{S}_x, \hat{S}_y + \{\pm\}$
 $\in \{\pm\}$
all in $\{\pm\}$ basis.

T TRICK! Ladder, raising & lowering $\hat{O}'s$!

↳ Hint! Whenever you see problems where eigenvalues are separated by exactly 1 unit of ang mom, (\hbar)
Then think of this trick!

ex. ^① Harmonic oscillations: $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$
 $n=0, 1, 2, \dots$

see Griss's &
my Q.M. I notes:

② Angular momentum: Chpt 6 Shermer!

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③ SPIN?

well sure! Spin is an intrinsic ang moment $\notin m_s$ separated by $1\hbar$ So YEAH!

The Ladder trick to get $\hat{S}_x \pm \hat{S}_y$ from $\hat{S}_z \pm \hat{S}^2$

Define $\hat{S}_+ = \hat{S}_x + i\hat{S}_y$

[in
+2 bases!]

$$\hat{S}_- = \hat{S}_x - i\hat{S}_y$$

will be able to get $\hat{S}_+ \pm \hat{S}_-$ in terms
of $\hat{S}_z \pm \hat{S}^2$

\therefore Finally get $\hat{S}_x \pm \hat{S}_y$

Here goes!

For now, take for granted that

$$\hat{S}_- |S_{m_s}\rangle \subset |S, m_{s-1}\rangle$$

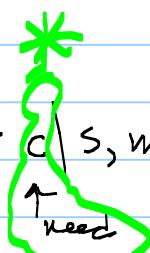
ie \hat{S}_- lowers
the proj of S_{spin}
by 1

$$\{ \hat{S}_+ |S_{m_s}\rangle \subset |S, m_{s+1}\rangle$$

ie \hat{S}_+ on $|S_{m_s}\rangle$
makes new eigenvector
 $|S, m_{s+1}\rangle$ that is a
state of S again
but one less in
projection

Same \longrightarrow

So, $\hat{S}_- |S_{m_s}\rangle = c |S, m_{s-1}\rangle$



Note: $(\hat{S}_-)^+ = \hat{S}_x + i\hat{S}_y = \hat{S}_+$ so

$$(\hat{S}_+)^+ = \hat{S}_x - i\hat{S}_y = \hat{S}_-$$

$$\hat{S}_- |s, m_s\rangle = C |s, m_s - 1\rangle$$

$\frac{1}{\sqrt{2}} \langle s, m_s | \hat{S}_+ \rangle$

so this $\left[\begin{array}{c} \hat{S}_- |s, m_s\rangle \\ \langle s, m_s | \hat{S}_+ \rangle \end{array} \right]$ Transposed, *

so $\langle s, m_s | \hat{S}_+ = \langle s, m_s - 1 | C^* =$ just
bra-space
rep of

OK... then...

$$\langle s, m_s | S_+ S_- | s, m_s \rangle = C C^* \underbrace{\langle s, m_s - 1 | s, m_s - 1 \rangle}_{=1}$$

$$= |C|^2 \text{ good.}$$

Now

$$S_+ S_- = (S_x + i S_y)(S_x - i S_y)$$

$$= S_x^2 + S_y^2 - i S_x S_y + i S_y S_x$$

recognize as

$$= S_x^2 + S_y^2 - i [S_x, S_y]$$

But we know this!

$$= S_x^2 + S_y^2 - i (i k S_z)$$

$$\hat{S}_+ \hat{S}_- = S_x^2 + S_y^2 - i(S_z S_2)$$

note $S^2 = S_x^2 + S_y^2 + S_z^2$

$$\hat{S}_+ \hat{S}_- = S^2 - S_z^2 + i S_z$$

WOW-AH! we've got \hat{S}_+ & \hat{S}_-
in terms of things

we

know & have eigenvalues

$$S_m, S^2 \& S_z$$

since

$$[S^2, S_z] = 0$$

∴

$$\langle S_m | \hat{S}_+ \hat{S}_- | S_m \rangle$$

$$\langle S_m | S^2 - S_z^2 + i S_z | S_m \rangle = C^2$$

$$= (S(S+1)h^2 - \frac{1}{2}m_s^2 + i h m_s) \langle S_m | S_m \rangle = C^2$$

$= 1$

so

$$C = \sqrt{\text{all that}} = \sqrt{S(S+1) - m_s(m_s+1)}$$

WOW-AH again!

$$S_- | S_m \rangle = \sqrt{S(S+1) - m_s(m_s-1)} | S_{m-1} \rangle$$

Now we know $\hat{S}_- |s_{m_s}\rangle = c_1 |s, m_s-1\rangle$

(ie what \hat{S}_- does when acting on $|+\rangle$ basis)

next we'll get

$$\hat{S}_+ |s_{m_s}\rangle = c_2 |s, m_s+1\rangle$$

we'll figure that out

So that we can get

$$S_- = S_x - iS_y$$

$$S_+ = S_x + iS_y$$

#1 add them $S_- + S_+ = 2S_x$, $S_x = \frac{1}{2}(S_+ + S_-)$

#2 sub them $S_- - S_+ = -2iS_y$; $S_y = \frac{1}{2i}(S_+ - S_-)$

we know
what S_+ & S_-
do on $|+\rangle$

\therefore we have S_x & S_y
in $|+\rangle$ basis!

H.W. Side Note: Get $S_+ |S, m_s\rangle = C_2 |S, m_s + 1\rangle$

As did S_x or S_z ,

$$\text{relog } S_+^+ = (S_x + iS_y)^+ = (S_x - iS_y) = S_-^-$$

$$S_0 \quad \left\{ S_+ |S, m_s\rangle = C_2 |S, m_s + 1\rangle \right\}^+$$

$$\left\langle S, m_s \right| S_- = \left\langle S, m_s + 1 \right| C_2^*$$

OK Then ...

$$\left\langle S, m_s \right| S_- S_+ |S, m_s\rangle =$$

$$\left\langle S, m_s + 1 \right| C_2^* C_2 |S, m_s + 1\rangle = C_2^* C_2$$

OK

$$\underbrace{\left\langle S, m_s \right| S_- S_+ |S, m_s\rangle}_{(S_x - iS_y)(S_x + iS_y)} = C_2^* C_2$$

$$(S_x - iS_y)(S_x + iS_y)$$

$$S_x^2 + S_y^2 + iS_xS_y - iS_yS_x \implies$$

so

$$S_z^2 - S_x^2 + i[S_x, S_y]$$

$$S_z^2 - S_x^2 + i[i\hbar S_z]$$

$$S_z^2 - S_x^2 - \hbar S_z$$

Note: $S^2 = S_x^2 + S_y^2 + S_z^2$
 $\{S_x, S_y\} = i\hbar S_z$

OK

$$\langle s, m_s | (s^2 - s_z^2 - \lambda s_z) | s, m_s \rangle = |c_2|^2$$

$$s(s+1)\lambda^2 - \lambda^2 m_s^2 - \lambda \lambda m_s \langle s, m_s | s, m_s \rangle = |c_2|^2$$

so

$$c_2 = \lambda \sqrt{s(s+1) - m_s(m_s+1)}$$

OK!

$$\hat{S}_+ |s, m_s \rangle = \lambda \sqrt{s(s+1) - m_s(m_s+1)} |s, m_s + 1\rangle$$

again

\hat{S}_+ operator on $|s, m_s\rangle$ & you
you need eigenvector of
some S_z , but one higher
in proj of S .

Back to here: we want $\hat{S}_x \downarrow \hat{S}_y$ from $\hat{S}_+ \downarrow \hat{S}_-$
 but
 Let's look @ our results 1st to
 see if all is cool

Found

$$\hat{S}_- | s m_s \rangle = \hbar \sqrt{s(s+1) - m_s(m_s-1)} | s, m_s-1 \rangle$$

$$\hat{S}_+ | s m_s \rangle = \hbar \sqrt{s(s+1) - m_s(m_s+1)} | s, m_s+1 \rangle$$

Check!

$$\text{Spin } \frac{1}{2} \Rightarrow \begin{array}{ccc} m_s = \frac{1}{2} & \rightarrow & |\frac{1}{2}, \frac{1}{2} \rangle \\ \text{---} & & = |s, m_s \rangle \\ m_s = -\frac{1}{2} & \text{---} & |\frac{1}{2}, -\frac{1}{2} \rangle \end{array}$$

OK:

$$1) \quad \hat{S}_- |\frac{1}{2}, \frac{1}{2} \rangle = \hbar \sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)} | \frac{1}{2}, -\frac{1}{2} \rangle$$

$$= \hbar \sqrt{\frac{3}{4} + \frac{1}{4}} = \hbar \sqrt{\frac{4}{4}} = \hbar | \frac{1}{2}, -\frac{1}{2} \rangle$$

$$\hat{S}_- |\frac{1}{2}, \frac{1}{2} \rangle = \lambda | \frac{1}{2}, -\frac{1}{2} \rangle \Leftrightarrow \hat{S}_- |\uparrow \rangle = \hbar | \downarrow \rangle$$

$$2) \quad \hat{S}_- |\frac{1}{2}, -\frac{1}{2} \rangle = \hbar \sqrt{\underbrace{\frac{1}{2}(\frac{1}{2}+1)}_{\frac{3}{4}} - \underbrace{(-\frac{1}{2})(-\frac{1}{2}-1)}_{-\frac{3}{4}}} | \frac{1}{2}, -\frac{3}{2} \rangle$$

$$= \hbar \sqrt{\frac{3}{4} - \frac{3}{4}} = 0$$

YEAH... Cool, there is no $-\frac{3}{2}$ proj of $S = \frac{1}{2}$

$$3) \quad \hat{S}_+ |\frac{1}{2}, \frac{1}{2} \rangle = \hbar \sqrt{\frac{3}{4} - (-\frac{1}{4})} | \frac{1}{2}, \frac{1}{2} \rangle = \hbar | \frac{1}{2}, \frac{1}{2} \rangle \Rightarrow \hat{S}_+ |\uparrow \rangle = \hbar |\uparrow \rangle$$

$$4) \quad \hat{S}_+ |\frac{1}{2}, -\frac{1}{2} \rangle = \hbar \sqrt{\frac{3}{4} - \frac{3}{4}} | \frac{1}{2}, -\frac{1}{2} \rangle = 0 \quad \text{again, no } +\frac{3}{2} \text{ proj of } S_{\text{spin }} \frac{1}{2}$$

OK: BALD to mission ... Find \hat{S}_x & \hat{S}_y
 ↳ eigen vectors & values
 IN $|+\rangle$ basis.

OK: here's where we are:

$$\text{1) } \hat{S}_- = \hat{S}_x - i\hat{S}_y$$

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y$$

$$\text{just go! } \hat{S}_- |s, m_s\rangle = C_1 |s, m_s-1\rangle$$

$$\& \hat{S}_+ |s, m_s\rangle = C_2 |s, m_s+1\rangle$$

so we know what \hat{S}_- & \hat{S}_+ do in $|+\rangle$ basis.

$$\text{OK not add 1) } \hat{S}_x = \frac{1}{2} (\hat{S}_+ + \hat{S}_-) \quad \text{sub 1)}$$

$$\hat{S}_y = \frac{1}{2i} (\hat{S}_+ - \hat{S}_-) \quad \text{sub 1)}$$

so we've got it!

recall

$$\hat{S}_x = \begin{pmatrix} \langle z | S_x | z \rangle & \langle z | S_x | -z \rangle \\ \langle -z | S_x | z \rangle & \langle -z | S_x | -z \rangle \end{pmatrix} \quad \text{sub}$$

$$\& \hat{S}_y = \begin{pmatrix} \langle S_y | S_y | z \rangle & \langle S_y | S_y | -z \rangle \\ \langle z | S_y | S_y \rangle & \langle -z | S_y | S_y \rangle \end{pmatrix}$$

so need to know matrix elements
 of \hat{S}_+ & \hat{S}_-

matrix elements of \hat{S}_x & \hat{S}_-

$$\hat{S}_x = \begin{pmatrix} \langle z | S_x | z \rangle & \langle z | S_x | -z \rangle \\ \langle -z | S_x | z \rangle & \langle -z | S_x | -z \rangle \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \hbar \langle z | z \rangle \\ 0 & +\hbar \langle -z | z \rangle \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix}$$

OK $\hat{S}_- = \begin{pmatrix} \langle z | S_- | z \rangle & \langle z | S_- | -z \rangle \\ \langle -z | S_- | z \rangle & \langle -z | S_- | -z \rangle \end{pmatrix}$

$$= \begin{pmatrix} -\hbar \langle z | z \rangle & 0 \\ +\hbar \langle -z | -z \rangle & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 \\ \hbar & 0 \end{pmatrix}$$

Now: $\hat{S}_x = \frac{1}{2}(S_+ + S_-) = \frac{1}{2} \begin{pmatrix} 0 & \hbar \\ \hbar & 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
in S_z basis

$\hat{S}_y = \frac{1}{2i}(S_+ - S_-) = \frac{1}{2i} \begin{pmatrix} 0 & +\hbar \\ -\hbar & 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$
in S_z basis

So... let's summarize: Spin $\hat{\vec{S}}$ in $| \pm \frac{1}{2} \rangle$ basis

$$\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

now: Dirac Equation

$$(c \vec{\alpha} \cdot \vec{p} + \beta m_0 c^2) \Psi = i \hbar \frac{d \Psi}{dt}$$

↑ requires solns

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \& \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(called)
recognized = Pauli Spin Matrices

$\sigma_z, \sigma_x, \sigma_y$ respectively

s_0

$$\hat{S}_z = \frac{\hbar}{2} \sigma_z, \quad \hat{S}_x = \frac{\hbar}{2} \sigma_x, \quad \hat{S}_y = \frac{\hbar}{2} \sigma_y$$

Nice & what we wanted But keep in mind what it means for Dirac Equation

Ψ must be include SPIN! SPIN is intrinsically inside relativity (ie Dirac equat)

Ψ_{Dirac} (ie w/ spin) \equiv SPINORS!