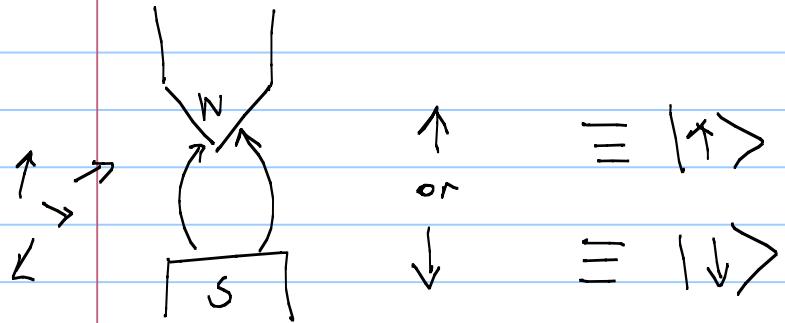


From S.G. experiment we see that space quantization of spin \Rightarrow 's 2 states only!



{ if we actually measure the spin angular momentum we get

$$\hat{S}_z |\uparrow\rangle = +\frac{\hbar}{2} |\uparrow\rangle$$

$$\hat{S}_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle$$

In other words, WE HAVE SOLVED The eigenvalue problem experimentally!

RECALL

Q.M.

Matrix

Schrödinger Wave Formalism

$$\text{solve } \hat{H} \psi_n = E_n \psi_n$$

Just an eigen problem

For ψ_n = eigen functions, E_n = eigen values

goal is to solve for $\psi_n \notin E_n$

Then **KEY** ψ_n 's = Complete Basis

Set to Build all possible solutions

Matrix Version For Spin, the experiment solves it for you!

$$\hat{H}_{\text{spin}} | \psi_{\text{spin}} \rangle = c | \psi_{\text{spin}} \rangle$$

$$| \psi_s \rangle = \text{eigen } \underline{\text{vectors}} = | \uparrow \rangle \text{ or } | \downarrow \rangle$$

$$\text{and } c \text{ is } \underline{\text{eigen values}} = \pm \hbar/2$$

Which is here the complete Basis

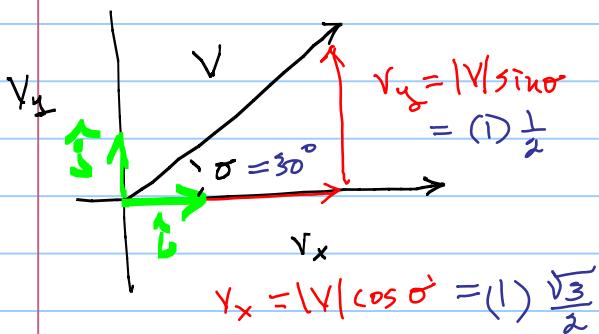
= Very Finite sized (2-D)

Once you have complete finite sized basis, easy to draw it's to vector space

and can really build the matrix formalism
of QM

So let's do it:

Take \vec{V} , $|V|=1$, in 2-D = Vector Space example



$$\vec{V} = V_x \hat{i} + V_y \hat{j}$$

Let \hat{i}, \hat{j} = Basis (STATES)

$$\left. \begin{array}{l} \vec{V} = \text{Some general state} \\ \text{Linear comb of } \hat{i}, \hat{j} \end{array} \right\} \quad \vec{V} = c_1 \hat{i} + c_2 \hat{j}$$

c_1, c_2 = projection coefficients.

Can ask what is probability of finding
 \vec{V} in \hat{i}, \hat{j} states:

$$\left. \begin{array}{l} \hat{i}; \frac{\sqrt{3}}{2} \\ \hat{j}; \frac{1}{2} \end{array} \right\} \text{no way} \quad \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{1+\sqrt{3}}{2} \neq 1$$

$$\left. \begin{array}{l} \hat{i}; (\frac{\sqrt{3}}{2})^2 = \frac{3}{4} \\ \hat{j}; (\frac{1}{2})^2 = \frac{1}{4} \end{array} \right\} \frac{3}{4} + \frac{1}{4} = 1 \text{ cool!} \quad \text{But no surprise}$$

because $|\vec{V}| = \sqrt{\vec{V} \cdot \vec{V}} = \sqrt{(c_1 \hat{i} + c_2 \hat{j}) \cdot (c_1 \hat{i} + c_2 \hat{j})}$

$$= \sqrt{c_1^2 + c_2^2} = 1$$

$$c_1^2 + c_2^2 = 1$$

Once we have a complete basis, Then
any vector

$$\vec{V} = c_1 \hat{i} + c_2 \hat{j}$$

\downarrow
 c_1, c_2 are normalized, Then

$$c_1^2 + c_2^2 = 1 \quad \text{identifying } c_i^2 = \begin{matrix} \text{prob of} \\ \text{being in} \\ \text{state } i \end{matrix}$$

$$c_j^2 = \begin{matrix} \text{prob of} \\ \text{being in} \\ \text{state } j \end{matrix}$$

So the (projections)² = probabilities

To find: given \vec{V} , what is c_1 ?

For the $\hat{i} \hat{j}$ Basis

Must specify as could have Any
basis we like -- here \hat{i}, \hat{j} = convenient

$$\hat{i} \cdot \vec{V} = \hat{i} \cdot (c_1 \hat{i} + c_2 \hat{j}) = c_1$$

$$\hat{j} \cdot \vec{V} = \hat{j} \cdot (c_1 \hat{i} + c_2 \hat{j}) = c_2$$

So prob's = $|(\hat{i} \cdot \vec{V})|^2$ in Vector Space (VS)

Further:

Since

$$\vec{v} = c_1 \hat{i} + c_2 \hat{j}$$

$$c_1 = \frac{\hat{i} \cdot \vec{v}}{\hat{i} \cdot \hat{i}}$$

$$c_2 = \frac{\hat{j} \cdot \vec{v}}{\hat{j} \cdot \hat{j}}$$

$$\vec{v} = (\hat{i} \cdot \vec{v}) \hat{i} + (\hat{j} \cdot \vec{v}) \hat{j}$$

$$\vec{v} = \hat{i} (\hat{i} \cdot \vec{v}) + \hat{j} (\hat{j} \cdot \vec{v})$$

Things, notation, a bit messy so
make change

$$\hat{i} = \hat{e}_i \quad \& \quad \hat{j} = \hat{e}_j$$

$$\vec{v} = \hat{e}_i (\hat{e}_i \cdot \vec{v}) + \hat{e}_j (\hat{e}_j \cdot \vec{v})$$

OR

$$\vec{v} = \sum_n (\hat{e}_n) (\hat{e}_n \cdot \vec{v})$$

$$n = i, j$$

OR

$$\vec{v} = \sum_{n=i,j} (\hat{e}_n) (\hat{e}_n) \cdot \vec{v}$$

≡ projection $\overset{\wedge}{\text{O}}$

OK, a bit awkward, but handy in a minute

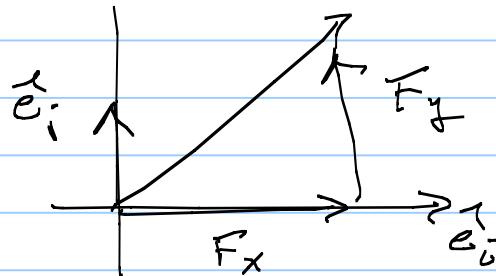
-- here's a check!

$$\text{claim } \vec{F} = \sum_{i,j} (\vec{e}_i)(\vec{e}_j) \cdot \vec{F}$$

$$= \vec{e}_i \cdot (\vec{e}_i \cdot [F_x \vec{e}_i + F_y \vec{e}_i]) + \vec{e}_i \cdot (\vec{e}_j \cdot [F_x \vec{e}_i + F_y \vec{e}_i])$$
$$= \vec{e}_i \cdot (F_x + 0) + \vec{e}_i \cdot (0 + F_y)$$

$$\vec{F} = F_x \vec{e}_i + F_y \vec{e}_i$$

yeah!



so

$$\vec{F} = \sum_{n=i,j} (\vec{e}_i)(\vec{e}_j) \cdot \vec{F} = F_x \vec{e}_i + F_y \vec{e}_j$$

Full Basis

This projection \uparrow doesn't change \vec{F}

really Except that it looks different now, it is a projection representation. But a change of representation ^{only} really is like multiplying by 1

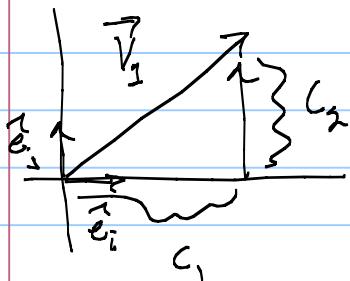
so $\mathbb{I} \equiv \text{identity operator} \equiv \sum_{\text{complete basis}} (\vec{e}_n)(\vec{e}_n)$.

OK, now make
Matrix Q.M.

Vector Space
say: \hat{e}_i, \hat{e}_j Basis

Then

$$\vec{V}_{\text{general}} = c_1 \hat{e}_i + c_2 \hat{e}_j$$



$$|\vec{V}_g|^2 = \text{normalized}$$

$$|c_1|^2 = |(\hat{e}_i \cdot \vec{V}_g)|^2 = \underset{\text{prob in state } \hat{e}_i}{\text{prob in state } \vec{V}_g}$$

$$|c_2|^2 = |(\hat{e}_j \cdot \vec{V}_g)|^2 = \propto ||\hat{e}_j||$$

$$\vec{V}_g = \sum_{n=1}^N [\hat{e}_n] (\hat{e}_n) \cdot \vec{V}_g$$

w/ identity \vec{I}

$$\vec{I} = \sum_{\text{complete basis}} [(\hat{e}_n) (\hat{e}_n)] -$$

Complex Abstract Vector Space
CAVS

say work in a $|\uparrow\rangle, |\downarrow\rangle$
space for spin

Then $|\psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle$
= generalized
state But don't call it a vector call it a ket!

$|\uparrow\rangle \neq |\downarrow\rangle$ form
complete basis KETS

$$\text{or } |\uparrow\rangle \quad |\psi\rangle \quad |\downarrow\rangle \quad c_1 \quad c_2$$

in some abstract spin space

$|\psi\rangle^2 = \text{normalized then}$

$$|c_1|^2 = \left| \begin{array}{l} \text{dot product of basis} \\ |\uparrow\rangle \text{ with general} \\ \text{state } |\psi\rangle \end{array} \right|^2$$

Huh? invert procedure:

$$|\uparrow\rangle \cdot |\psi\rangle \equiv \langle \uparrow | \psi \rangle$$

(vector dot product)

where we've created a
"bra" space $\langle \uparrow |$ so that

$$|\uparrow\rangle \cdot |\psi\rangle = \underbrace{\langle \uparrow | \psi \rangle}$$

"bra" "ket" = bra ket!

dot product = bra ket!

$$\text{so now } |G|^2 = |\langle \uparrow|\psi \rangle|^2$$

Now let's not get too lost...
is rewrite orbit in terms
of good old vectors
Then

$$c_1^2 = \left| \underbrace{\langle \hat{L}_i | \vec{F} \rangle}_{\text{just dot product}} \right|^2$$

$$c_1^2 = |F_x|^2$$

$$\rightarrow \text{then } |c_2|^2 = |\langle \hat{L}_j | \psi \rangle|^2$$

Further

$$|\psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle$$

any general
ket

in $|\uparrow\rangle, |\downarrow\rangle$ basis

$$\text{recall } \mathbb{I} = \sum_{\text{Basis}} [\hat{e}_n] [\hat{e}_n]$$

rewritte w/ basis $\not\in$ ket's

$$\mathbb{I} = \sum_{\text{Basis}} [\hat{e}_n] [\hat{e}_n]$$

if Basis = $|\uparrow\rangle \not\in |\downarrow\rangle$ then

$$\mathbb{I} = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| \quad \dots$$

lets see

$$|\psi\rangle = \sum |\psi\rangle$$

$$= (\uparrow\langle\uparrow| + \downarrow\langle\downarrow|) |\psi\rangle$$

$$= |\uparrow\rangle\langle\uparrow|\psi\rangle + |\downarrow\rangle\langle\downarrow|\psi\rangle$$

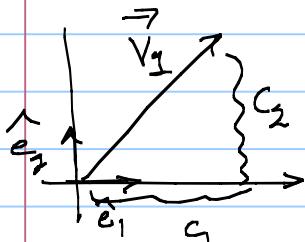
$$|\psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle$$

Just like $\vec{F} = \sum \vec{F} = \left(\sum_{\text{basis}} [\vec{e}_n] \cdot [\vec{e}_n]^\top \right) \vec{F} = c_1 \hat{i} + c_2 \hat{j}$

Now to Matrices!

Vector Space to Matrices

$$\vec{V}_g = c_1 \hat{e}_1 + c_2 \hat{e}_2$$



Say $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\hat{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(column matrices)

NOTE:
1 row
or
1 column matrix
= vector

$\therefore \text{so } \vec{V}_g = (c_1, c_2) \text{ in } \hat{e}_1, \hat{e}_2 \text{ basis.}$

Now natural extension of dot product ---

$$\hat{e}_1 \cdot \hat{e}_1 = (1, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1+0=1$$

$$\hat{e}_1 \cdot \hat{e}_2 = (1, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0+0=0$$

$\therefore \text{so } c_1 = \hat{e}_1 \cdot \vec{V}$

$$= (1, 0) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1$$

KEY: dot prod yields projections!

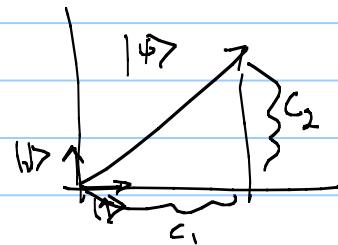
c_1 = how much of \vec{V} is projected onto \hat{e}_1

$\therefore \text{then } |c_1|^2 = ((\hat{e}_1 \cdot \vec{V}) / |\hat{e}_1|)^2 = \text{prob of finding } \vec{V} \text{ in state } \hat{e}_1$

CAVS to Matrices

$$|\psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle$$

* note c_1 & c_2 might be, can be complex (i)



{ identifying basis

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore |\psi\rangle = (c_1, c_2)$$

Now natural extension of "braket"

$$\langle \uparrow | \uparrow \rangle = (1, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$\langle \uparrow | \downarrow \rangle = (1, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

if read: is in $|\uparrow\rangle$, prob of being in $|\uparrow\rangle = 1$.

is in state $|\downarrow\rangle$ prob of being in state $|\uparrow\rangle = 0$

$$c_1 = \langle \uparrow | \psi \rangle$$

$$= (1, 0) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1$$

So Braket Plays same role w/ same interpretation

What's more is now have a basis for how to "construct" bras from kets!

$|Kets\rangle$ = good basis set

or

general state $(|4\rangle)$ you build from a complete basis.

Now we know idea of dot product in this representation is

$$\psi \cdot \psi \sim \langle \psi | \psi \rangle$$

where we need to build the bra $\langle \psi |$

OK: so

$$|\psi\rangle = C_1 |\psi\rangle + C_2 |\psi\rangle \text{ say}$$

where $C_1 \notin C_2$ might be complex;

Then

$$\langle \psi | \psi \rangle \text{ must} = 1$$

so need

$$(C_1, C_2) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 1$$

{
ie proj on itself
when normalized
must = 1, ie
prob of being
yourself = 1}

$$C_1^2 + C_2^2 = 1 = \text{True if } C_1 \notin C_2 = \text{real
but is complex
need}$$

$$C_1^* C_1 + C_2^* C_2 = 1$$

which
suggests

$$\langle \psi | \psi \rangle = 1$$

$$(c_1^*, c_2^*) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 1$$

so, identify bra w/ ket as
so follows:

$$\langle \psi | = \left[(\psi) \text{ Transposed} \right]^*$$

Transposed means for Matrix

$$M_{ij},$$

$$(M_{ij})^T = M_{ji}$$

so

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix}$$

i.e interchange rows
columns.

$$\text{so } \left[\begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} \right]^T = (c_{11}, c_{21})$$

$$\therefore \left[\begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} \right]^{+*} (c_{11}^*, c_{21}^*) = \langle \psi |$$

Recap:

Instead of solving we did experiment, SG

$$\hat{S}_z \psi_{spin} = S_z \psi_{spin}$$

$\frac{1}{2}$ sound solns

$$|\psi_{spin}\rangle = |\uparrow\rangle, |\downarrow\rangle$$

in Zndr
= complete

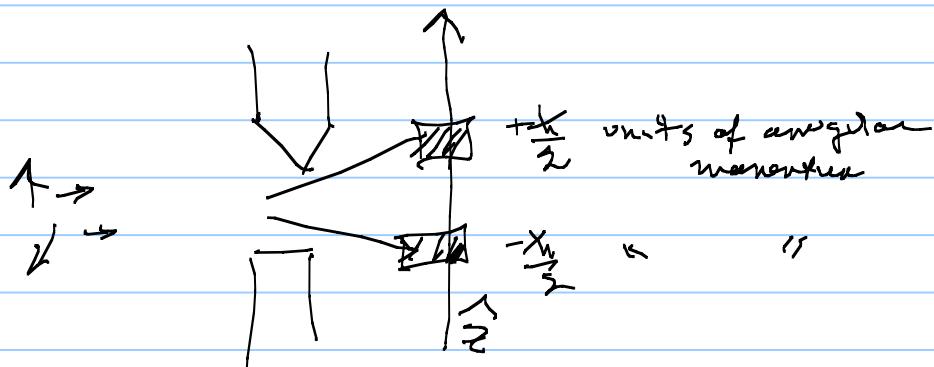
Basis

$$\text{w/ eigen values } \pm \frac{\hbar}{2}$$

$$\therefore \hat{S}_z |\uparrow\rangle = +\frac{\hbar}{2} |\uparrow\rangle$$

$$\hat{S}_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle$$

all for



We say: Space Quantization in \mathbb{R}^3 dir leads to good complete spin basis $|\uparrow\rangle, |\downarrow\rangle$

$$\therefore |\psi_{spin}\rangle = C_1 |\uparrow\rangle + C_2 |\downarrow\rangle$$

C_1, C_2 must be complex

$$C_1 = \langle \uparrow | \psi_g \rangle = \text{prob of } \psi_g \text{ in } |\uparrow\rangle$$

$$\therefore |C_1|^2 = |\langle \uparrow | \psi_g \rangle|^2 \approx \text{prob of finding particle in state } |\uparrow\rangle \text{ when in } |\psi_g\rangle$$

$$\hat{1} = |\uparrow\downarrow\rangle\langle\uparrow\downarrow|$$

very handy!

insert anywhere
cause it is
essentially
multiplying by 1