

To be able to do Q.M. need:

1) Special Relativity = idea that laws of physics must be invariant to all inertial frames

* Kinematics (constant accel)

* Maxwell's equations

⇒ space-time coords } space +
 $\vec{x} \rightarrow x^\mu \equiv (ct, \vec{x})$ } time
mix

and

Electric & Magnetic field mixing

$\vec{E} \& \vec{B} \rightarrow F^{\mu\nu} \equiv$ Electromagnetic Stress Tensor

$$= \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & B_x & 0 \end{pmatrix}$$

$x_\mu x^\mu = c^2 t^2 - |\vec{x}|^2 = \text{invariant scalar}$

idea: some frames see space, others time but what doesn't change is $|x^\mu|$

Both invariant to Lorentz Transformations

* Dynamics! (ie cons of \vec{p})

new def of $\vec{p} = m\vec{v} \rightarrow \gamma m\vec{v}$
 & Energy & momentum mix to keep invariant

$\vec{p} \rightarrow p^\mu = (E/c, \vec{p})$

$$P_{\mu} P^{\mu} = \frac{E^2}{c^2} - |\vec{p}|^2 = \text{invariant scalar}$$

$$= \left(\frac{p^2 c^2 - m_0^2 c^4}{c^2} \right) - p^2 = m_0^2 c^2$$

Follows that $E_T^2 = p^2 c^2 - m_0^2 c^4 = \gamma m_0 c^2$

1) Set $p = 0$

$$E_T = m_0 c^2$$

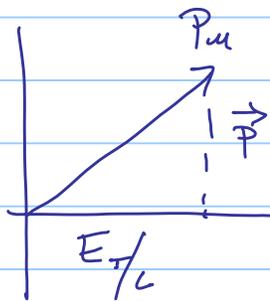
2) Set $m_0 = 0$

$$p = \frac{E_T}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} = \frac{h}{\lambda} \frac{2\pi}{2\pi} \frac{1}{\lambda}$$

a) $p = \hbar k$

b) $\lambda = \frac{h}{p} = \text{De Broglie 1924}$

extended idea that a) & b) hold for non $m_0 = 0$ particles too!



$$P_{\mu} P^{\mu} = \frac{E^2}{c^2} - |\vec{p}|^2 = m_0^2 c^2$$

means some observers see

energy some see momentum

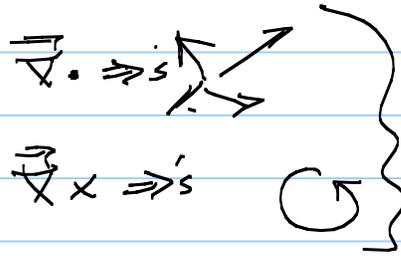
↳ this, $E_T^2 = p^2 c^2 - m_0^2 c^4$

Q.F.T built from all space-time, X^{μ} energy-momentum, P^{μ} coordinates

to fully incorporate SR from start,

2.) Maxwell's Equations (write & understand on own = requirement) Physics

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \frac{\rho_{ext}}{\epsilon_0} + 0 \\ \vec{\nabla} \times \vec{E} &= 0 + -\frac{d\vec{B}}{dt} \\ \vec{\nabla} \cdot \vec{B} &= 0 + 0 \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}\end{aligned}$$



Thus all about \vec{E} & \vec{B}

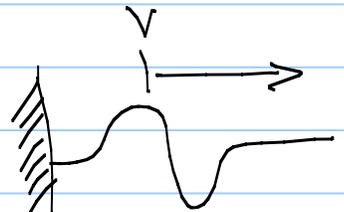
* magnetic monopole terms add symm but change nothing so are possible. would they go? Instanton makes all = 0

OK, $\nabla \times \left(\begin{matrix} \nabla \times \vec{E} \\ \text{or} \\ \nabla \times \vec{B} \end{matrix} \right) \downarrow \vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$\Rightarrow s$

$$\left. \begin{aligned}\frac{d^2 \vec{E}}{dx^2} &= \mu_0 \epsilon_0 \frac{d^2 \vec{E}}{dt^2} \\ \frac{d^2 \vec{B}}{dx^2} &= \mu_0 \epsilon_0 \frac{d^2 \vec{B}}{dt^2}\end{aligned} \right\}$$

recognize



traveling wave on string

$$\frac{d^2 s}{dx^2} = \frac{1}{v^2} \frac{d^2 s}{dt^2}$$

need in R.Q.F.T

\rightarrow so this = plane traveling waves

$$\begin{aligned}\vec{E}(x,t) &= \vec{E}_0 e^{\pm i(kx - \omega t)} \\ \vec{B}(x,t) &= \frac{E_0}{c} e^{\pm i(kx - \omega t)}\end{aligned}$$

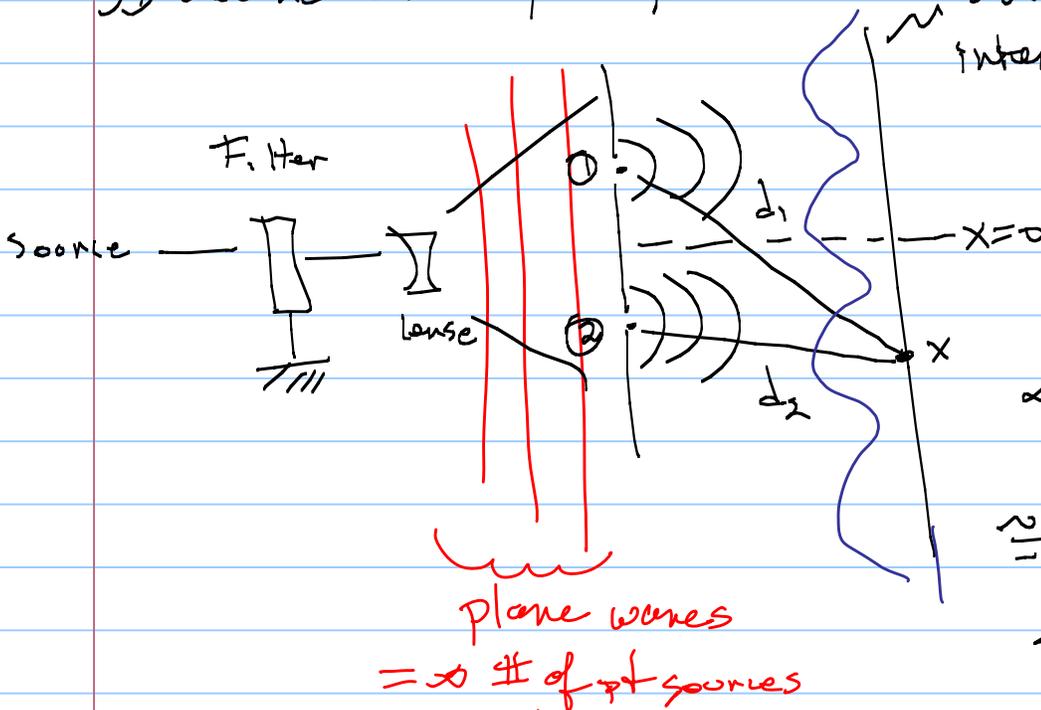
* $v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$

* hold x fixed: wave repeats in $1 \text{ full } \pi$

so $\omega \pi = 2\pi$ or $\omega = \frac{2\pi}{\pi}$

* hold t fixed: wave must repeat in $x \Rightarrow$ so $kx = 2\pi$; $k = \frac{2\pi}{\pi}$ wave #.

3) Double Slit w/ Light



$\left. \begin{matrix} \text{Huygens} \\ \text{princ.} \end{matrix} \right\} = \text{Huygens princ.}$

using plane wave solns ① & ② interfere

$$I \propto E^2 = E_1^2 + E_2^2 + \underbrace{2 E_1 E_2 \cos k(d_2 - d_1)}_{\text{interference term}}$$

NICE Argument!

LIGHT = good old wave. BUT let filter attenuate until 1 click @ a time!

Get same pattern eventually BUT conclude:

* 1 click @ time \Rightarrow particle = photon.

* Can't know where the "1" particle ends up

* But after a while get normal intensity pattern (ie "1" particle needs then to interfere w/ itself)

* \therefore Prob has to be $\propto |E^2|$ then must be that $\int_{x=-\infty}^{\infty} |E^2| dx = 1$ so density

$|E^2| = \frac{\text{prob}}{\text{length}} = \text{prob density}$

so \uparrow Light = wave
 e^- = particle
 1900

\downarrow double slit + 1923 Compton Scattering

+ 1905 Photo-electric

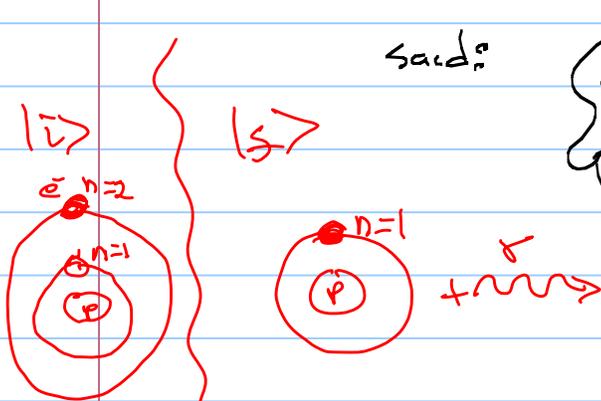
(turns out not to be
 definitive case for
 Light = particles, can explain
 using wave only)

\Rightarrow

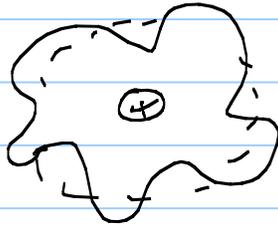
Light = wave + particle

so Light e^- = particle + $\left\{ \begin{matrix} \text{wave} \\ \text{particle} \\ \text{wave} \\ \text{particle} \end{matrix} \right\}$

1912 \Rightarrow Bohr Model of Atom to Solve
 Atomic Spectra Problems



said:



e^- too wave
 wave-like
 w/ quantized
 orbital angular
 momentum.

$$L = h\hbar = \vec{r} \times \vec{p}$$

$L_n = 0, 1, 2 \dots$ (quantized)

$$E_i = E_f + \gamma$$

$$\gamma = E_i - E_f = \frac{-13.6 \text{ eV}}{n^2} \left(\frac{1}{n_i^2} - \frac{1}{n_f^2} \right)$$

Get H-atom spectra
 Solve for

= + Energies

$$E_n = \frac{-13.6 \text{ eV}}{n^2}$$

to get spectra
 energies!

HUGE! = old - QM.
 no idea why all worked.

2 eV \pm about 1 eV
 = visible

1912 - Old-QM. Bohr Model

1925 Heisenberg (Hans Foren)

1924 DeBroglie

notes: 4 vector
S.R. invariants
 X^4, X^k ; $X^4 = (ct, \vec{x})$
 P^4, P^k ; $P^4 = (E/c, \vec{p})$

Jordan + Born

Dirac

Schrödinger by suggestion of P. Debye "waves" $e^{i(\omega t + \vec{k} \cdot \vec{x})}$ = invariant phase so

$$K^4 \equiv \left(\frac{\omega}{c}, \vec{k} \right)$$

P. Debye "waves" The wave equat? $K^k X^k = \left(\frac{\omega}{c}, \vec{k} \right) \cdot (ct, \vec{x}) = (\omega t - \vec{k} \cdot \vec{x})$

$$K^k X^k = \left(\frac{\omega}{c}, \vec{k} \right) \cdot (ct, \vec{x}) = (\omega t - \vec{k} \cdot \vec{x})$$

Matrix Version of Q.M

1926

M.Q.M

?

1926

W.Q.M

P^4 invar is $K^4 X^k$ so relation between maybe P^4 and K^4 ?

Schröd 1926 $\nabla^2 \psi = E \psi$

M.Q.M.

W.Q.M

square integral solns (ie $\int |\psi|^2 dx = \text{finite}$)

\Rightarrow 's Hilbert Vector Space

\Rightarrow 's arbs like Vector space

\therefore treat w/ lin Alg

\Rightarrow ie Matrices!

$P^4 \propto K^4$
 $\Rightarrow P^4 = \hbar K^4 = \left(\frac{h\nu}{c}, \hbar \vec{k} \right)$
it works! $\uparrow \left(\frac{E}{c}, \vec{p} \right)$

$\vec{p} = \hbar \vec{k}$ which is all true

is $m=0$



Born 1926; $\Psi = \text{prob amplitude} = ?$

but $|\Psi|^2 = \text{prob density!}$

physically $E^2 = p^2 c^2 + m_0^2 c^4$
 $p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} = \hbar k$

So he



Dirac 1929: Schröd + S.R = Dirac Equat

ventured most the case for $m \neq 0$



Feynman, Tomonaga, Schwinger, 1963

$\Psi = \text{particle waves}$

$E = \text{Force Fieldware}$

= Q.F.T.

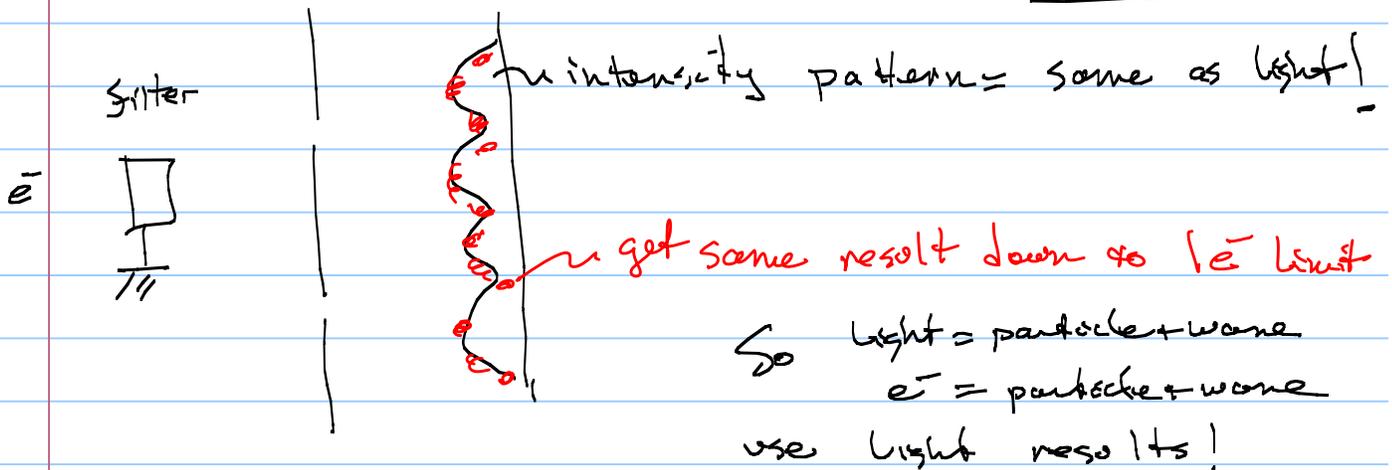
unified w/ X^4, P^4 (ie SR)

$$\lambda = \frac{h}{p} \text{ for } m=0$$

mass or no mass particles!

Nobel's 1932: Heisenberg
 1933: Dirac & Schrödinger
 1954: Born
 1965: Feynman, Tomonaga, Schwinger.

e^- s = waves ξ $|\Psi|^2 = \text{prob?}$ Double-Slit!



Light
 wave equation: From Maxwell, solve E

intensity: $\propto |E|^2$

prob: $\propto |E|^2$

normalized $\int_{-\infty}^{+\infty} |E|^2 dx = 1$

$\therefore |E|^2 = \text{prob density}$

sols $\propto e^{i(kx - \omega t)}$

e^-
 From Schröd, solve Ψ

$\propto |\Psi|^2$

$\propto |\Psi|^2$

$\int_{-\infty}^{+\infty} |\Psi|^2 dx = 1$

$|\Psi|^2 = \text{prob density}$

$\Psi = \text{prob amplitude}$

sols $\Psi \propto e^{i(kx - \omega t)}$

traveling plane wave

Q.M.

(particle like) $\bar{\Psi} = \frac{1}{\hbar} \int \Psi$ (wave-like) Ψ $\omega) \Psi \propto e^{i(kx - \omega t)}$

Schröd

$$\left(\frac{p^2}{2m} + V\right) \bar{\Psi} = \hbar\omega(\bar{\Psi})$$

since $p = \hbar k$
to get from Ψ
need $-i\hbar \frac{\partial}{\partial x} = \hat{p}$
so
 $p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$

since $E = \hbar\omega$
to get from Ψ
 $i\hbar \frac{\partial}{\partial t}$

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V\right) \bar{\Psi} = i\hbar \frac{\partial}{\partial t} \bar{\Psi}$$

Schröd time dep Equat

Dirac: look

$$E_{tot} = \sqrt{p^2 c^2 + m_0^2 c^4}$$

so maybe

$$(c\vec{\alpha} \cdot \vec{p} + \beta m_0 c^2) \bar{\Psi} = i\hbar \frac{\partial \bar{\Psi}}{\partial t}$$

The Dirac Equation!

is solve,
 $\vec{\alpha}$ = matrix
which requires
3 - Pauli Spin
Matrices!
ie: SR Requires
existence of Spin!

Spin must be
put in by hand into Schröd

Schro' Equation

$$\hat{H} \Psi = i\hbar \frac{\partial \Psi}{\partial t} = \text{time dep.}$$

* Look for separable solus } good idea: easier
to form complete basis
to build other solus.

$$\Psi = \zeta(t) \psi(x)$$

$$\zeta(t) = e^{-\frac{iE}{\hbar}t}$$

∴ $\zeta(x)$ part separates to

$$\hat{H} \psi(x) = E \psi(x) = \text{Schro' time indep soln!}$$

Solve this for $\psi(x)$

∴ soln to time depend prob

$$\Psi(x,t) = e^{-\frac{iE}{\hbar}t} \psi(x) \text{ ALWAYS!}$$

So solve always in practice for

$\psi(x) = \text{time indep solus!}$

Problem is that
prob $\propto |\Psi(x,t)|^2 \neq \zeta(t)$
so how do you get dynamics?
Not to worry for now

OK (cause 1.) time dep = easy $e^{-\frac{iE}{\hbar}t}$

2.) $\Psi = \text{complete basis! solus.}$

3.) ∴ Can easily build time depend

This is an eigenvalue problem!

OK

$$\hat{H}\psi = E\psi$$

\Rightarrow 's $\psi =$ eigenfunction, $\psi(x)$
or
eigenvector, $|\psi\rangle$

sols $\psi(x)$ are
eigenfunctions of \hat{H} w/
eigenvalues E

Idea of eigen vector:

$\hat{H} \equiv$ operator acts on $\psi(x)$

say $\hat{H} \equiv \frac{d}{dx}$

if $\psi_1(x) = x^2$

then $\hat{H}\psi_1 = \frac{d}{dx}(x^2) = 2x$

so $\hat{H}\psi_1 = (\text{constant})\psi_1$

\hat{H} changes ψ_1

if $\psi_2(x) = e^{ikx}$

then

$\hat{H}\psi_2 = \frac{d}{dx} e^{ikx} = ike^{ikx}$

or

$\hat{H}\psi_2 = (ik)\psi_2$

Same idea w/
rotations!

in
general

$$M\vec{r} = \vec{R}$$

ie operate on
vector \vec{r} & get
new vector \vec{R}

But for

$$M\vec{r} = c\vec{r}$$

$$M\vec{r} = c\vec{r}$$

so M

changes length
of \vec{r} but
not its angle!

Conclude: eigen functions are not altered

by being operated on but do spit out
eigenvalues

so eigenfunctions 'contain' eigenvalue info about the Q.M. states, ψ .

If \hat{O} 's are Hermitian, $\left[\hat{O}^\dagger \equiv (\hat{O}^T)^* \right] = \hat{O}$

Then the eigenvalues are REAL #'s

ξ are observable properties of the state ψ .

$$\text{so } \hat{H}\psi(x) = E\psi(x)$$

Hilbert, square integrable,
complete function space for
continuous variable

Finite vector space

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$\hat{H}\psi = E\psi : \psi'' + A\psi = 0$$

solved in \mathbb{R}

$$(\hat{H} - E)|\psi\rangle = 0$$

$$\text{solve } \det|\hat{H} - E| = 0$$

$$\Psi_{\text{TOT}} = e^{\frac{-E_n t}{\hbar}} \psi(x)$$

$$\Psi_{\text{TOT}} = e^{\frac{-E_n t}{\hbar}} |\psi\rangle$$



use whichever is most convenient.

ex: Spin: clearly Finite space \uparrow or \downarrow

$$\text{so } |\psi\rangle = |\uparrow\rangle \text{ or } |\downarrow\rangle$$

or H-atom: well not so clear
but
here is deal.

if want to know what H-atom
looks like, you'll need

$$|\psi(x)|^2 \text{ so solve } \hat{H}\psi = E\psi.$$

But if just want to compute energies
& transitions, you don't need to know ψ
just

$$|\psi\rangle = |n, l, m\rangle$$

$$\hat{H}\psi = E\psi$$



$\langle \hat{O} \rangle =$ expectation of $\hat{O} =$ Energy or position or ...

$$= \int \psi^* \hat{O} \psi dx = \int \hat{O} \psi^* \psi dx$$

pulls out answer, say $c = \#$

$$= \int \psi^* c \psi dx$$

$$= \int c \psi^* \psi dx$$

$$= \int c (\text{Prob}(x))$$

= Average!

recall: $\langle \hat{O} \rangle = \sum_i s_i P_i$
from basic stats

$$\hat{H}|\psi\rangle = E|\psi\rangle$$



$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = \text{kets}$$

Then $\langle \psi| = (|\psi\rangle)^\dagger$

$$\langle \psi| = (a^*, b^*) = \text{bra}$$

$$\hat{H} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

so

$$\langle \hat{O} \rangle = \langle \psi| \hat{O} |\psi\rangle = \langle \hat{O} \psi | \psi \rangle$$

Also keys:

always report $\bar{x} \pm \Delta x$

standard deviation

can show

$$\text{std } x = \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \quad \text{so}$$

Final answer

$$\int \psi^* \hat{O} \psi dx \pm \sqrt{\int \psi^* \hat{O}^2 \psi dx - (\int \psi^* \hat{O} \psi dx)^2}$$

Then idea of commuting \hat{O} 's

$$[\hat{H}, \hat{A}] = 0 \Rightarrow \hat{H}\hat{A}\psi = \hat{A}\hat{H}\psi$$

ψ is simultaneous eigenfunction for

both \hat{H} & \hat{A}

i.e. that neither \hat{H} nor \hat{A} changes ψ , thus eigenvalues of both \hat{H} & \hat{A} are in ψ can be extracted w/o problem

or if $[\hat{H}, \hat{A}] = 0$ then \Rightarrow 's $\hat{H}\psi = h\psi$ ↖ eigenvalue of \hat{H}
 $\&$ $\hat{A}\psi = a\psi$ ↖ eigenvalue of \hat{A}

$\Rightarrow \hat{H}\hat{A}\psi = \hat{H}(a\psi) = ha\psi$

a & b are commuting eigenvalues of ψ
 They are exact linear attributes of the state ψ — on both can be measured simultaneously!

On the contrary: if

$$[\hat{H}, \hat{B}] \neq 0 \Rightarrow \hat{H}\hat{B}\psi - \hat{B}\hat{H}\psi \neq 0$$

means that either

$$\hat{H}\psi \neq h\psi$$

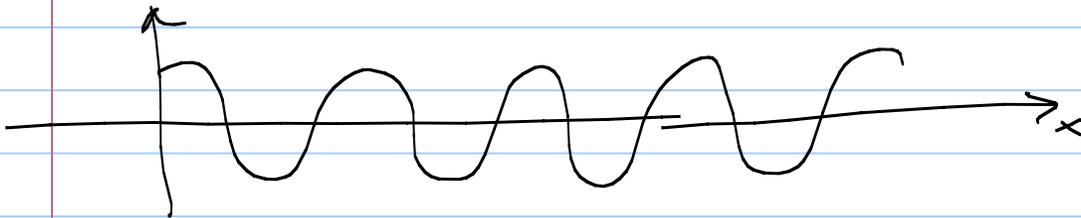
or

$$\hat{B}\psi \neq b\psi$$

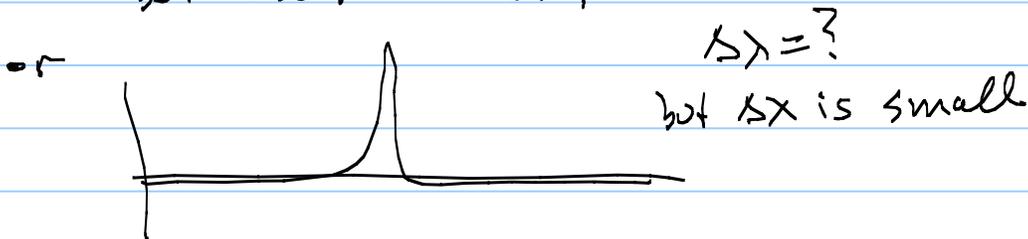
or that either (both) \hat{H} & \hat{B} destroy (change) ψ .

\hat{H} & \hat{B} are not commuting eigenvalues & clearly then cannot be measured simultaneously!

it all stems from ψ 's = wave-like
as with any classical wave even



$\Delta \lambda =$ uncertainty in λ is small
but what is Δx ?



\therefore Cannot know both simultaneously!

in

Q.M. $\hat{X} = x$
 $\hat{P} = -i\hbar \frac{\partial}{\partial x}$

so from

$$\Delta X \sim \Delta x$$

$$\hat{P} = \hbar k = \frac{2\pi\hbar}{\lambda} \therefore \Delta P \sim \Delta \lambda$$

$$\Delta X \Delta P \geq \hbar/2$$

that $[\hat{X}, \hat{P}] \neq 0$

in fact $[\hat{X}, \hat{P}] = -i\hbar$

Scene w/ $E \ll t$

ΔT - small
 ΔE - ?

ΔE - small
 ΔT - ?

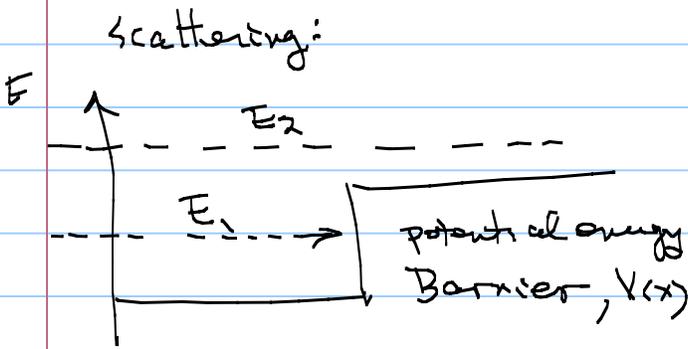
$E = \hbar \omega$ so $\Delta E \ll \hbar \omega$ so $\Delta E \Delta t \geq \hbar/2$
 $= \hbar \frac{1}{\pi}$

From Formalism to Problems:

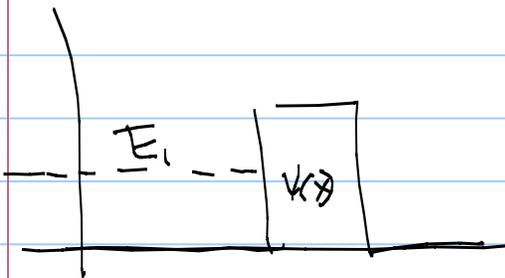
$$\hat{H}\psi = E\psi$$

I: Free particle, scattering states

we considered



⚡ tunneling



⚡ get ψ incident

ψ reflected

ψ transmitted

or

$R = \% \text{ reflected}$

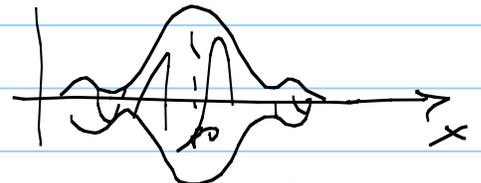
$T = \% \text{ transmitted}$

continuous energies

if don't watch, can use sum of $\rightarrow + \leftarrow$ plane waves $\psi_1 + \psi_2$ of different energy

But if watch, need to use uncertainty princ Δx , now need uncertainty

in $\Delta p \Rightarrow$ energy so need to construct wave packet = sum of plane waves whose momenta are centered around p_0



$|\Delta x|$

localizes in Δx

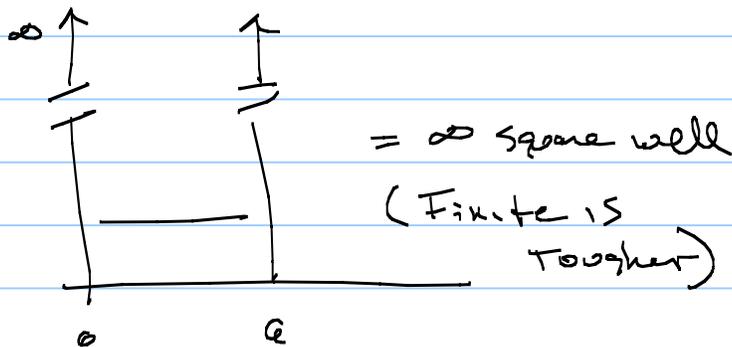
@ expense of

$\Delta p \Rightarrow$ need add

band of other

energies!

II Bound States \rightarrow Discrete energy states!



$$\psi_n \sim \sin \frac{1}{2} \text{ \& } \cos \frac{1}{2}$$

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2m a^2}$$

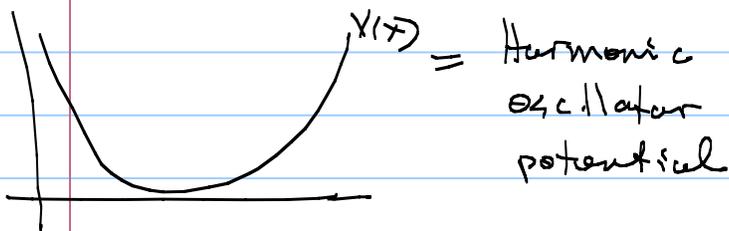
$$n = 1, 2, 3, \dots$$

$n \neq 0$ so $n=1$
ground lowest
state for particle

is it is there!

$n=0$, no particle

Energy levels \neq
evenly spaced!



ψ_n are Hermite polynomials!

$$E_n = \left(n + \frac{1}{2}\right) \hbar \omega$$

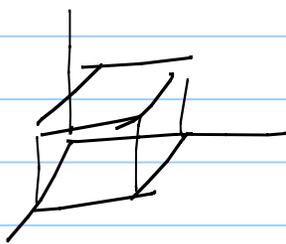
? $n=0$ OK! For a particle

But notice, that lowest energy is $\neq 0$
it is $\frac{1}{2} \hbar \omega$? why? is no energy, would

\Rightarrow particle at exact bottom of well but this violates
uncertainty principle, thus all Harmonic Oscillators
never have zero energy.

Finally energy levels are evenly spaced! by units
of $\hbar \omega$. Thus H.O. Lumps in energy = quantized
ex photons & phonons = lumps in vibrational energy!

More bound states:



3-D
∞ well

$$\psi_{xyz} = \psi_x \psi_y \psi_z \quad \begin{matrix} \text{each} \\ \text{from 1-D} \\ \text{case} \end{matrix}$$

$$E_{n_x, n_y, n_z} = \left(\frac{\hbar^2 \pi^2}{2m a^2} \right) (n_x^2 + n_y^2 + n_z^2)$$

just like 3 separate
1-D problems

Then of course the H-atom: 3-D spherical
Coords

$$\psi(r, \sigma, \phi) = \frac{1}{\sqrt{4\pi a_0^3}} e^{-r/a_0}$$

$$\hat{H} \psi(r, \sigma, \phi) = E \psi(r, \sigma, \phi)$$

$$\text{need } \hat{H}(r, \sigma, \phi) = \frac{\hbar^2 \nabla^2}{2m} + \underbrace{V(r, \sigma, \phi)}$$

∴ get

∴ $V(r)$ only
always
get

$$\Psi(r, \sigma, \phi) = e^{-iEt/\hbar} \psi(r, \sigma, \phi) = e^{-iEt/\hbar} R(r) Y_{lm}(\sigma, \phi)$$

$Y_{lm}(\sigma, \phi)$
Spherical harmonics

$$E_n = \frac{-4e^4}{(4\pi\epsilon_0)^2 \hbar^2 n^2} \quad \frac{1}{n^2} = -13.6 \text{ eV} / n^2$$

* note, not $f(l, m_l)$

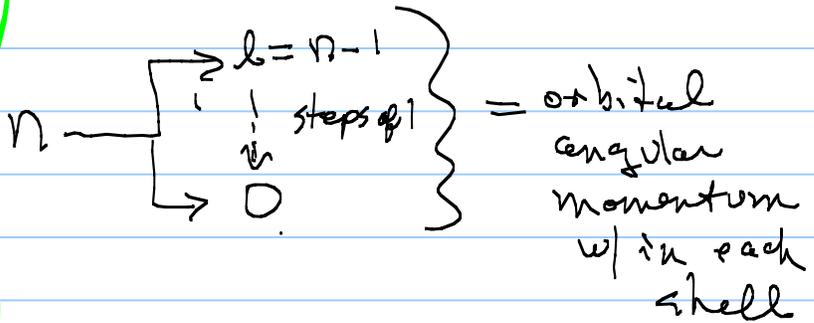
great!
same a
Bohr BUT
more!

So $\Psi = e^{-i\frac{E}{\hbar}t} \underbrace{|n, l, m_l\rangle}_{\text{sols only for}}$

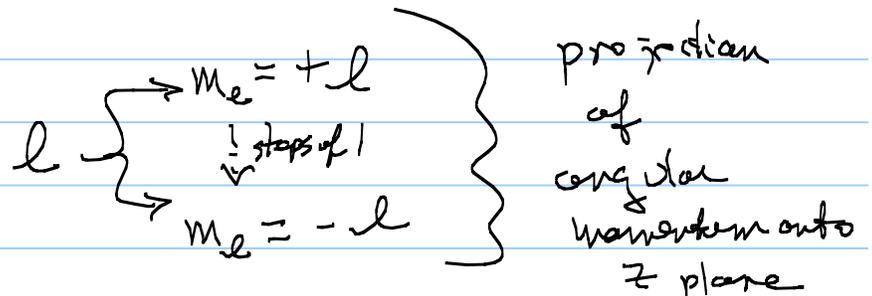
$n = 1, 2, 3, \dots$ = Principal Quantum # or shell

For $V(r, \theta, \phi) = S(r)$ only
 $\Psi_e^m(r, \theta, \phi) = \text{always soln.}$
 can't measure L_x, L_y, L_z simultaneously but can measure $|L^2|$ & L_z projection.
 So $L^2 \Psi_e^m = \hbar^2 l(l+1) \Psi_e^m$
 & $L_z \Psi_e^m = \hbar m_l \Psi_e^m$
 So $\Psi_e^m = \text{maximally informed state of angular momentum!}$

For given n



For given l

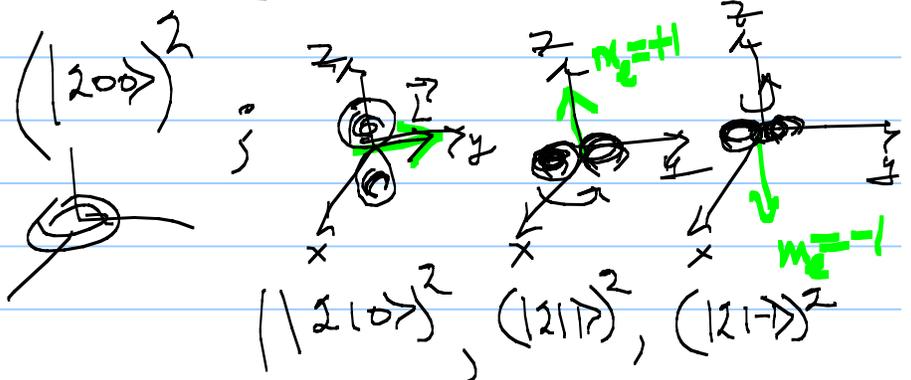
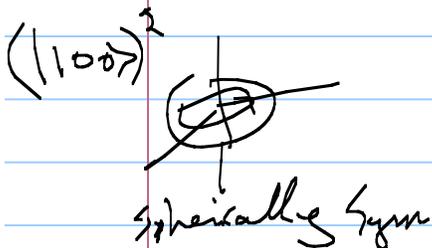


ex: H-atom

$n=1 \Rightarrow l=0, m_l=0$

$n=2 \Rightarrow l=1 \Rightarrow \sum m_l = 0$

$l=0 \Rightarrow m_l=0$



So we have

H-atom

$$\hat{H}\Psi = i\hbar \frac{d\Psi}{dt}$$

$$\Psi(\vec{r}, t) = e^{-i\frac{E_n}{\hbar}t} R(r) \sum_{\ell, m} \langle \sigma, \ell \rangle = e^{-i\frac{E_n}{\hbar}t} |n, \ell, m\rangle$$

Great... what about SPIN?

Cause Schröd is non relativistic,
Spin is
not "inside" of it

Note it is for Dirac equation

$$(c\vec{\alpha} \cdot \vec{p} + \beta mc^2)\Psi = i\hbar \frac{d\Psi}{dt}$$

$\vec{\alpha}$ requires Spin!

So we will now have to put spin in by hand!

$$\Psi = e^{-i\frac{E_{nb}}{\hbar}t} |n, \ell, m\rangle \langle \text{Spin} \rangle \quad \text{Opt 0}$$