E.F. Devaney BSC Physics: Quantum II

Note Title 3/28/2007

Sherrer Chpt 8 Spin H.W. 8.8 - 8-14

8.8 in an S_z basis, check commutation relations for $\vec{S}$ (matrix-rep)

* commutation relations are recovered!

$$\underset{\sim}{\vec{S}} = \frac{\hbar}{2}\underset{\sim}{\vec{\sigma}} \Rightarrow \underset{\sim}{S_x} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Pauli spin matrices

$$\underset{\sim}{S_y} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\underset{\sim}{S_z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So ....

$$S_x S_y = \frac{\hbar^2}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad S_y S_x = \frac{\hbar^2}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad S_z S_x = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$S_y S_z = \frac{\hbar^2}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad S_z S_y = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad S_x S_z = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

OK: $[S_x, S_y] = S_x S_y - S_y S_x = \frac{\hbar^2}{4}\left\{ i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

$$= \frac{\hbar^2}{4} 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar^2}{2} i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar S_z$$

$$[S_y, S_z] = S_y S_z - S_z S_y = \frac{i\hbar^2}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\hbar S_z$$

$$[S_z, S_x] = S_z S_x - S_x S_z = \frac{\hbar^2}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{\hbar^2}{i}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{2}(-i)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = i\hbar\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\hbar S_y$$

8.9 Spin 1/2 particles run thru SG.

$$|\psi_i\rangle \Longrightarrow \boxed{SG\,\hat{z}} \begin{cases} \longrightarrow |+z\rangle \Rightarrow \frac{9}{25}\,\% \\ \longrightarrow |-z\rangle \Rightarrow \frac{16}{25}\,\% \end{cases}$$

Let $|\psi_i\rangle = c_1|+z\rangle + c_2|-z\rangle$

Then $|\langle z|\psi_i\rangle|^2 = c_1^* c_1 = \frac{9}{25} \quad \therefore \; c_1 = 3/5$

& $|\langle -z|\psi_i\rangle|^2 = c_2^* c_2 = \frac{16}{25} \quad \therefore \; c_2 = 4/5$

a.) Conclude: one good $|\psi_i\rangle = \frac{3}{5}|z\rangle + \frac{4}{5}|-z\rangle$

is it normalized?

$$1 \overset{?}{=} \langle\psi_i|\psi_i\rangle = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \frac{9}{25} + \frac{16}{25} = 1$$

yes.

b.) How many solns are there?

WELL... claim $c_1^* c_1 = \frac{9}{25} \overset{\text{But}}{\Rightarrow} \underbrace{c_1 = e^{i\sigma_1}\frac{3}{5} \quad \sigma_1 = \text{any}!}_{\text{works Fine!}}$

& $c_2^* c_2 = \frac{16}{25} \overset{\text{Could be}}{\Longrightarrow} c_2 = e^{i\sigma_2}\frac{4}{5} \quad \sigma_2 = \text{any}!$

So ∞ # of solns where

$$|\psi_i\rangle = e^{i\sigma_1}\left(\frac{3}{5}\right)|\uparrow\rangle + e^{i\sigma_2}\left(\frac{4}{5}\right)|\downarrow\rangle \Longrightarrow$$

8.9 cont.

Now ... recall idea of convenient way to "reduce" your $|\psi\rangle$'s

$$= [\text{mag stuff}]\,[\text{common phase}]\begin{pmatrix}\psi_a \leftarrow \text{no phase} \\ \psi_b \leftarrow \text{relat to phase}\end{pmatrix}$$

so

lets see ...

$$|\psi_i\rangle = \begin{pmatrix} e^{i\sigma_1}\frac{3}{5} \\ e^{i\sigma_2}\frac{4}{5}\end{pmatrix}$$ ~ here factor out just phase

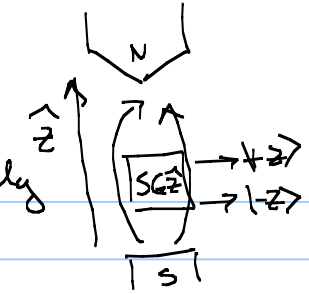
$$|\psi_i\rangle = e^{i\sigma_1}\begin{pmatrix}\frac{3}{5} \\ \frac{4}{5}e^{i(\sigma_2-\sigma_1)}\end{pmatrix}$$

never matters

relative phase does!

8.10

SG problem: normally



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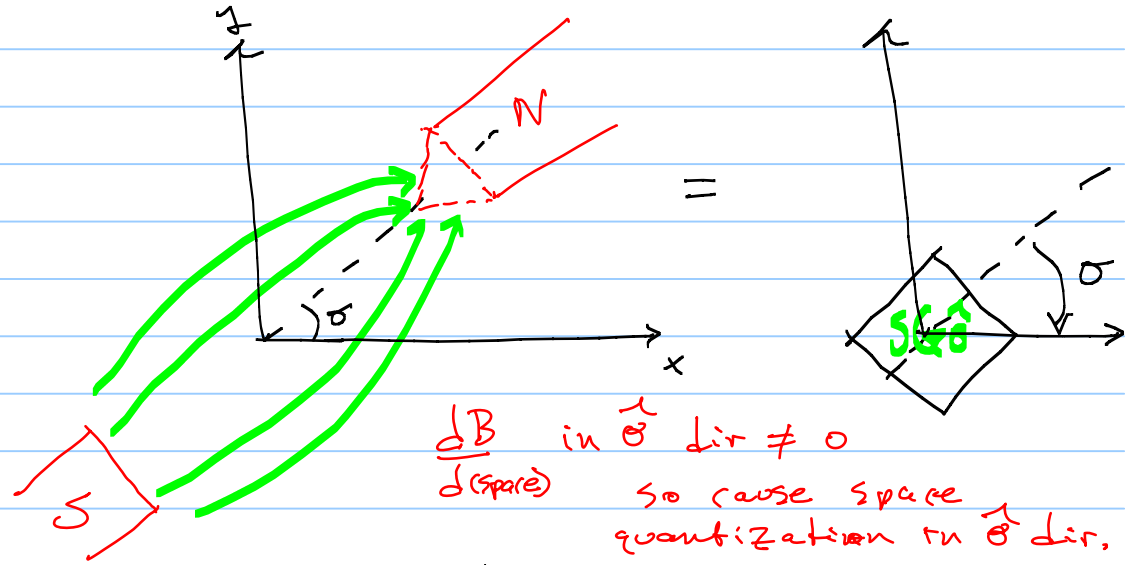


$\frac{dB}{d(\text{space})}$ in $\hat{\sigma}$ dir $\neq 0$

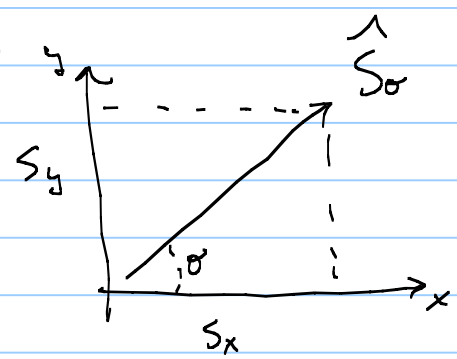
so cause space quantization in $\hat{\sigma}$ dir.

So, we have a $SG\hat{\sigma}$, σ = anywhere in xy plane

Well, messy but no huge deal since we know $\underset{\sim}{S}_x$, $\underset{\sim}{S}_y$ & $\underset{\sim}{S}_z$ in an S_z basis.

We now essentially ask, what is $\hat{\underset{\sim}{S}}_\sigma$ in an S_z basis.

Well



$$\hat{S}_\sigma = (\cos\sigma) S_x + (\sin\sigma) S_y$$

OK:

now we ask for soln to

$$\hat{S}_\sigma |\psi\rangle = c|\psi\rangle$$ = good old eigen problem!

means solve for eigenvalues 1st then eigenvectors

8.10 Cont

So solve $\hat{S}_\sigma |\psi\rangle = c|\psi\rangle$

do in S_z basis, matrix rep

$$\hat{S}_\sigma = \cos\sigma\, S_x + \sin\sigma\, S_y = (\cos\sigma)\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (\sin\sigma)\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\underset{\sim}{S}_\sigma = \frac{\hbar}{2}\begin{pmatrix} 0 & \cos\sigma - i\sin\sigma \\ \cos\sigma + i\sin\sigma & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i\sigma \\ e^{i\sigma} & 0 \end{pmatrix}$$

$\underset{\sim}{S}_\sigma |\psi\rangle = c|\psi\rangle \Rightarrow$ Find eigenvalues & then eigenvector for each.

$$(\underset{\sim}{S}_\sigma - c)|\psi\rangle = 0$$

$$(\underset{\sim}{S}_\sigma - \mathbb{I}\, c)|\psi\rangle = 0$$

only has non zero solns is det of secular equat $= 0$

$$\therefore \det\left|\underset{\sim}{S}_\sigma - \underset{\sim}{c}\right| = 0$$

$$\det\begin{vmatrix} -c & \frac{\hbar}{2}e^{-i\sigma} \\ \frac{\hbar}{2}e^{i\sigma} & -c \end{vmatrix} = 0$$

solving $c^2 - \left(\frac{\hbar}{2}\right)^2 \underbrace{(e^{-i\sigma}e^{i\sigma})}_{1} = 0$

$$c^2 = \frac{\hbar^2}{4}; \quad \boxed{c_1 = \frac{\hbar}{2},\ c_2 = -\frac{\hbar}{2}}$$

= our two eigenvalues

8.10 cont

OK Find eigenvectors for C_1 & C_2.

$C_1 = \frac{\hbar}{2}$

$$\hat{S}_{\sigma} |\psi\rangle = C_1 |\psi\rangle$$

$$\cancel{\frac{\hbar}{2}} \begin{pmatrix} 0 & e^{-i\sigma} \\ e^{i\sigma} & 0 \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \cancel{\frac{\hbar}{2}} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}$$

* $|\psi\rangle$ must be 2-component since $\hat{S}_{\sigma}$ is 2×2

OK... you factor to get 2 equations

1) $\psi_b e^{-i\sigma} = \psi_a$

2) $\psi_a e^{i\sigma} = \psi_b$

So, components are related (always happens like this)

Always try this: let $\psi_a = 1$

Then see what happens.

well (2) $\psi_b = (1) e^{i\sigma} = e^{i\sigma}$ OK

see if works back in (1)

$$e^{i\sigma} e^{-i\sigma} \stackrel{?}{=} \psi_a$$

$$1 = 1 \quad \text{yeah!}$$

OK so

The (A) eigenvector for $C_1 = \frac{\hbar}{2}$

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ e^{i\sigma} \end{pmatrix}, \text{ is in normal?}$$

$$\langle\psi_1|\psi_1\rangle = (1 \; e^{-i\sigma}) \begin{pmatrix} 1 \\ e^{i\sigma} \end{pmatrix} = 2; \text{ no! so}$$

8.10 cont

$$|\psi_1\rangle = \begin{pmatrix} N(1) \\ N(e^{i\sigma}) \end{pmatrix}$$

$$\langle\psi_1|\psi_1\rangle = N(1 \quad e^{-i\sigma})\begin{pmatrix} N \\ Ne^{i\sigma} \end{pmatrix} = 1$$

$$N^2\, 2 = 1$$

$$N = \frac{1}{\sqrt{2}}$$

Finally...

$$|\psi_1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}}(1) \\ \frac{1}{\sqrt{2}} e^{i\sigma} \end{pmatrix}$$

Back in explicit $|\pm z\rangle$ or $|\pm\uparrow\rangle$ Basis

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{e^{i\sigma}}{\sqrt{2}}|\downarrow\rangle$$

w/ $\hat{S}_\sigma|\psi_1\rangle = +\frac{\hbar}{2}|\psi_1\rangle$

now do same thing for 2nd eigenvalue

8.1D cont

$\underbrace{C_2 = -\frac{\hbar}{2}}_{\text{look for 2nd eigenvector } |\psi_2\rangle}$: $\hat{S}_{\vec{n}\sigma}|\psi_2\rangle = C_2|\psi_2\rangle$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\sigma} \\ e^{i\sigma} & 0 \end{pmatrix}\begin{pmatrix} \psi_c \\ \psi_d \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} \psi_c \\ \psi_d \end{pmatrix}$$

1.) $\psi_d e^{-i\sigma} = -\psi_c$

2.) $\psi_c e^{i\sigma} = -\psi_d$

try $\psi_c = 1$

Then (2) $\psi_d = (-1)(+1)e^{i\sigma} = -e^{i\sigma}$

check in (1) $\underbrace{-e^{i\sigma}e^{-i\sigma}}_{-1} = -\psi_c$

$-1 = -(1)$

yup!

so

$$|\psi_2\rangle = N\begin{pmatrix} 1 \\ -e^{i\sigma} \end{pmatrix}$$

Then $\langle\psi_2|\psi_2\rangle = N^2 \begin{pmatrix} 1 & -e^{-i\sigma} \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\sigma} \end{pmatrix} = 1$

$\Rightarrow$ is $N^2(2) = 1$

$N = \frac{1}{\sqrt{2}}$

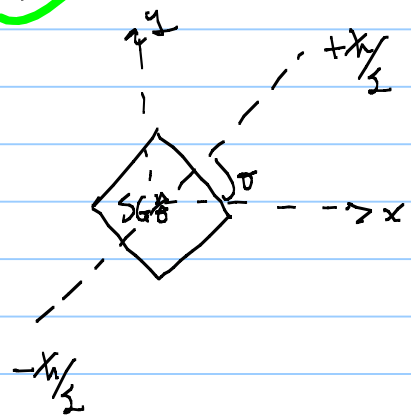
$$\therefore |\psi_2\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{-e^{i\sigma}}{\sqrt{2}}|\downarrow\rangle$$

where $\hat{S}_\sigma|\psi_2\rangle = -\frac{\hbar}{2}|\psi_2\rangle$

Continued.

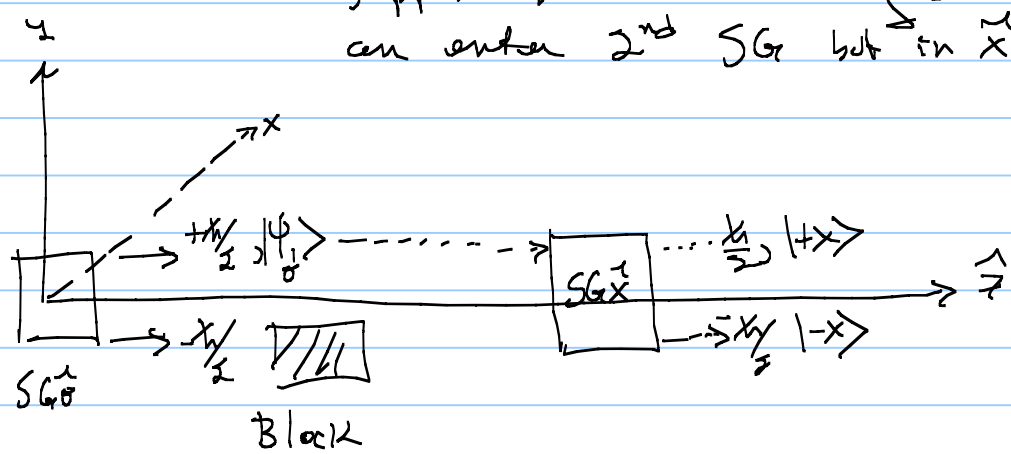
8.10 part (b)

Ok:



w/ particles traveling out of pg.

Suppose send these along $\hat{z}$ can enter 2nd SG but in $\hat{x}$



What is prob of measuring $-\frac{\hbar}{2}$, $|-x\rangle$?

well

$$|\psi_1\rangle \rightarrow \boxed{SG\hat{x}} \rightarrow \underset{\text{out}}{|-x\rangle}$$

So need proj of $|-x\rangle$ on $|\psi_1\rangle$ or

proj $= \langle -x|\psi_1\rangle$ then

$$\text{prob} = \left|\langle -x|\psi_1\rangle\right|^2$$

But will need everything in $|\pm z\rangle$ basis where we've defined $|-x\rangle$ & $|\psi_1\rangle$

So $\rightarrow$

8.10 contin

$$\text{prob} = |\langle -x | \psi_1(\sigma)\rangle|^2 \quad \text{where} \quad |-x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)$$

$$\& \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + e^{i\sigma}|\downarrow\rangle\right)$$

so

$$= \left|\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{e^{i\sigma}}{\sqrt{2}}\end{pmatrix}\right|^2$$

$$= \left|\frac{1}{2}\left(1 - e^{i\sigma}\right)\right|^2$$

$$= \frac{1}{4}\left(1 - e^{-i\sigma}\right)\left(1 - e^{i\sigma}\right)$$

$$= \frac{1}{4}\left(1 - e^{i\sigma} - e^{-i\sigma} + e^{-i\sigma}e^{i\sigma}\right)$$

$$= \frac{1}{4}\left(2 - \left(e^{i\sigma} + e^{-i\sigma}\right)\right)$$

$\hookrightarrow$ recall $e^{i\sigma} = \cos\sigma + i\sin\sigma$

$e^{-i\sigma} = \cos\sigma - i\sin\sigma$

so 1.) $e^{i\sigma} + e^{-i\sigma} = 2\cos\sigma$

2.) $e^{i\sigma} - e^{-i\sigma} = 2i\sin\sigma$

$$\text{prob} = \frac{2}{4}\left(1 - \cos\sigma\right)$$

$$\text{prob} = \frac{1}{2}\left(1 - \cos\sigma\right)$$

$\sim$ text says

$$\text{prob} = \sin^2\left(\sigma/2\right)$$

$\frac{1}{2}(1-\cos\sigma) \overset{?}{=} \sin^2(\sigma/2)$

$\sigma = 0$

$0 = 0$

$\sigma = \pi$

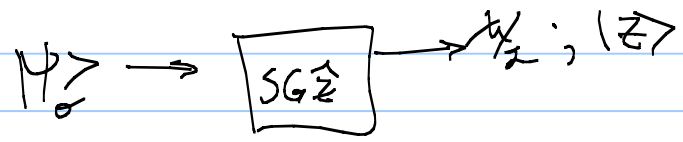
$1 = 1$

so I think so

I forget the trig identity

8.10 continued

b. ... continued what about ?



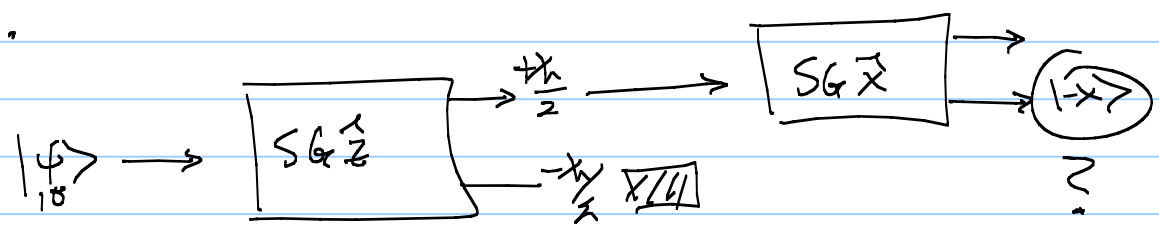
well

proj; $\langle \hat{z}|\psi\rangle$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{e^{i\sigma}}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\text{prob} = (\text{proj})^* (\text{proj}) = \frac{1}{2}$$

c.



proj; $\langle -x|+z\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$

$$\text{prob} = |\text{proj}|^2 = \frac{1}{2}$$

8-11 Spin 1/2 particle: $|\psi\rangle = \sqrt{\frac{2}{3}}\,|\uparrow\rangle + i\sqrt{\frac{1}{3}}\,|\downarrow\rangle$

a) normal?

$$\begin{pmatrix}\sqrt{\frac{2}{3}} & -i\sqrt{\frac{1}{3}}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{2}{3}} \\ i\sqrt{\frac{1}{3}}\end{pmatrix} = \frac{2}{3}+\frac{1}{3} = 1 \; ; \; \underbrace{|1|^2 = 1}_{\text{OK}}$$

b.) $|\psi\rangle \rightarrow \boxed{SG\hat{x}} - |-x\rangle$?

proj $= \langle -x|\psi\rangle$ all in $|\pm\uparrow\rangle$ basis

$$|-x\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{2}{3}} \\ i\sqrt{\frac{1}{3}}\end{pmatrix} = \frac{1}{\sqrt{3}} - i\frac{1}{\sqrt{6}}$$

Then

prob $|\text{proj}|^2 = (\text{proj}^*)(\text{proj})$

$$= \left(\frac{1}{\sqrt{3}} + i\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}} - i\frac{1}{\sqrt{6}}\right)$$

$$= \frac{1}{3} - \cancel{i\frac{1}{\sqrt{18}}} + \cancel{i\frac{1}{\sqrt{18}}} + \frac{1}{\sqrt{36}}$$

$$= \frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{1}{2}$$

c.) $|\psi\rangle \rightarrow \boxed{SG\hat{z}} \rightarrow -\frac{1}{2}, |-z\rangle \longrightarrow \boxed{SG\hat{x}} \rightarrow |+x\rangle$

well need proj $\langle x|-z\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}0 \\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}$

Then prob $= |\langle x|-z\rangle|^2 = \frac{1}{2}$

8.12 spin ½ e^- precessing in $\vec{B}_{ext}$. $\therefore \omega = \frac{2g}{2m} B_{ext}$

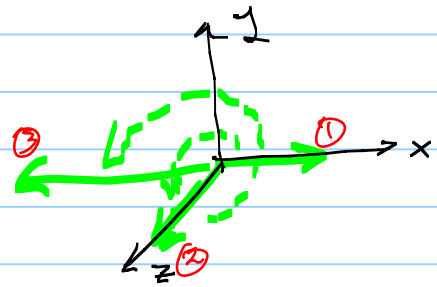
$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\omega t + \sin\omega t \\ \cos\omega t - \sin\omega t \end{pmatrix}$$

describe plane of rotation: $* \ \omega t = \pi = 2\pi$

① $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+x\rangle$

② $|\psi(t=\frac{\pi}{4\omega})\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+z\rangle$

③ $|\psi(t=\frac{\pi}{2\omega})\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-x\rangle$



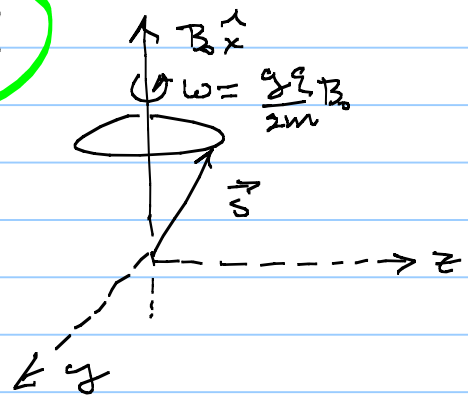
so e^- is in x-z plane

$\therefore \vec{B}_{ext} = B\hat{y}$

spinning counter clockwise as shown.

8-13 ---- see class notes!

8-14



@ $t=0$ $\quad |\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\omega t}{2}} \\ e^{i\frac{\omega t}{2}} \end{pmatrix}$

a) For $t>0$ compute prob of $\frac{\hbar}{2}$ $|+z\rangle$ dir!

So need proj of $|z\rangle$ on $|\psi\rangle$

$$|z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

proj: $\quad (1\ 0)\begin{pmatrix} \frac{e^{-i\frac{\omega t}{2}}}{\sqrt{2}} \\ \frac{e^{i\frac{\omega t}{2}}}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} e^{-i\frac{\omega t}{2}}$

prob $= (\text{proj})^*(\text{proj}) = \frac{1}{\sqrt{2}} e^{i\frac{\omega t}{2}} \frac{1}{\sqrt{2}} e^{-i\frac{\omega t}{2}} = \frac{1}{2} = 50\%/50\%$ $|\pm z\rangle$

b) How about $\hbar/2$ but in $|x\rangle$ dir?

$$|x\rangle = \frac{1}{\sqrt{2}}(\uparrow + \downarrow)$$

so

proj: $\quad \left(\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\right)\begin{pmatrix} \frac{e^{-i\frac{\omega t}{2}}}{\sqrt{2}} \\ \frac{e^{i\frac{\omega t}{2}}}{\sqrt{2}} \end{pmatrix} = \frac{1}{2}\left(e^{-i\frac{\omega t}{2}} + e^{i\frac{\omega t}{2}}\right)$

now $e^{i\sigma} = \cos\sigma + i\sin\sigma$

$$\underline{e^{-i\sigma} = \cos\sigma - i\sin\sigma}$$

$$e^{i\sigma} + e^{-i\sigma} = 2\cos\sigma$$

so

$$\text{proj} = \frac{1}{2}\left(2\cos\left(\frac{\omega t}{2}\right)\right) = \cos\left(\frac{\omega t}{2}\right)$$

$\therefore$

$$\text{prob} = \cos^2\left(\frac{\omega t}{2}\right) \text{ of finding in } |+x\rangle$$