

See: Townsend Chapter 2

In a CAVS such a spin, The space = 2-D
is Finite,

Thus

$$\mathbb{I} = \sum_{\text{space}} |i\rangle \langle i| =$$

$$|\uparrow\rangle \langle \uparrow| + |\downarrow\rangle \langle \downarrow|$$

= complete.

For S.G.

$$\begin{array}{c} \uparrow \\ \hline \downarrow \end{array} \quad \begin{array}{c} \uparrow+z & +\frac{\hbar}{2} \\ \hline \downarrow-z & -\frac{\hbar}{2} \end{array}$$

$$\begin{aligned} w | \quad \sum_z \left| \uparrow \right\rangle &= +\frac{\hbar}{2} \left| \uparrow \right\rangle \\ \sum_z \left| \downarrow \right\rangle &= -\frac{\hbar}{2} \left| \downarrow \right\rangle \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{ie } \pm |z\rangle \text{ basis}$$

we say the $|kets\rangle$ & eigenvalues for the "z"
basis are complete.

Now: Lets move from CAVS to Matrix Rep.

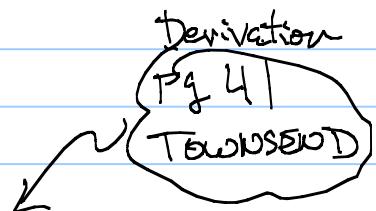
$$|z\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |-\bar{z}\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

* note
 z = quantized
 space dir.
 Arbitrary of
 course but
 by convention.

The ? now is, how do we get the matrix representations of our \hat{S}^z 's?

We are interested in

$$\hat{A}|4\rangle = E|4\rangle$$



but The math is more general

Start

$$A|4\rangle = |4\rangle$$

(ie not nec eigenvector)

lets agree to work in a complete basis

equiv to saying chose (cartes, cyl or sphr basis)

or spin basis!

$$|\uparrow\rangle = |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$$

$$|\downarrow\rangle = |\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow|$$

As long as it is complete \hat{A} will be all that there is

* But is will be "Matrix rep of \hat{A} in $|\pm 1s_z\rangle$ basis say"

So:

$$|4\rangle = \sum_{z\text{-basis}} |4\rangle = |\downarrow\rangle\langle z|4\rangle + |\uparrow\rangle\langle z|4\rangle$$

$$|\downarrow\rangle\langle z|4\rangle - |\uparrow\rangle\langle z|4\rangle = |\downarrow\rangle\langle z|4\rangle + |\uparrow\rangle\langle z|4\rangle$$

Then $\hat{A}|\psi\rangle = |\psi\rangle$ becomes

$$\hat{A} \left\{ |z\rangle\langle z| + |-z\rangle\langle -z| \right\} = \left\{ |z\rangle\langle z| + |-z\rangle\langle -z| \right\}$$

Quick reminder of ordinary vector space!

$$\vec{F} = m\vec{a}$$

$$(F_x \hat{i} + F_y \hat{j}) = (m a_x \hat{i} + m a_y \hat{j})$$

try $\hat{i} \cdot (\vec{F} = m\vec{a}) \Rightarrow F_x = m a_x$

then $\hat{j} \cdot (\vec{F} = m\vec{a}) \Rightarrow F_y = m a_y$

so

$$\vec{F} = m\vec{a} \Rightarrow \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} m a_x \\ m a_y \end{pmatrix} = 2 \text{-separate eqns,}$$

1 for each D obtained w/
dot prod
= projection

so let's do the same here!

Since Basis = $|z\rangle$

we want to do prod of w/ $(\hat{A}|\psi\rangle = |\psi\rangle)$

so we want $\langle \pm z | \hat{A} |\psi\rangle = \langle \psi |$

or

$$\langle +z | \hat{A} \{ |z\rangle \langle z|\psi\rangle + |-z\rangle \langle -z|\psi\rangle \} = \{ |z\rangle \langle z|\psi\rangle + |-z\rangle \langle -z|\psi\rangle \}$$

$$\{ \langle -z | \quad " \quad " \quad]$$

or

$$\langle z | \hat{A} | z \rangle \langle z | \psi \rangle + \langle z | \hat{A} | -z \rangle \langle -z | \psi \rangle = \langle z | z \rangle \langle z | \psi \rangle + \langle z | -z \rangle \langle -z | \psi \rangle$$

$$\langle -z | \hat{A} | z \rangle \langle z | \psi \rangle + \langle -z | \hat{A} | -z \rangle \langle -z | \psi \rangle = \langle -z | z \rangle \langle z | \psi \rangle + \langle -z | -z \rangle \langle -z | \psi \rangle$$

recalling $\langle z | z \rangle = \langle -z | -z \rangle = 1$

$$\langle -z | z \rangle = \langle z | -z \rangle = 0$$

OR

$$\begin{pmatrix} \langle z | \hat{A} | z \rangle & \langle z | \hat{A} | -z \rangle \\ \langle -z | \hat{A} | z \rangle & \langle -z | \hat{A} | -z \rangle \end{pmatrix} \begin{pmatrix} \langle z | \psi \rangle \\ \langle -z | \psi \rangle \end{pmatrix} = \begin{pmatrix} \langle z | \psi \rangle \\ \langle -z | \psi \rangle \end{pmatrix}$$

Ok: now:

$$|\psi\rangle_z = |z\rangle\langle z|\psi\rangle + |-z\rangle\langle -z|\psi\rangle$$

$$= \langle z|\psi\rangle(|z\rangle) + \langle -z|\psi\rangle(|-z\rangle)$$

write $\vec{z} = F_x \hat{i} + F_y \hat{j}$

projections onto basis

and as long as we all agree on a bases, just write

$$\vec{z} = (F_x, F_y) \text{ i.e}$$

as the projections!

so $|\psi\rangle_{z\text{-basis}} = (\langle z|\psi\rangle, \langle -z|\psi\rangle)$

make it a ket = column vector

$$|\psi\rangle_z = \begin{pmatrix} \langle z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$$

Similarly: $|\psi\rangle_z = \langle z|\psi\rangle(|z\rangle) + \langle -z|\psi\rangle(|-z\rangle)$

or

$$|\psi\rangle_z = \begin{pmatrix} \langle z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$$

so we have

$$\hat{A}|\psi\rangle = |\psi\rangle$$

\downarrow once we all agree dealing
w/ $|\pm z\rangle$ basis

$$\begin{pmatrix} \langle z|\hat{A}|z\rangle & \langle z|\hat{A}|-\bar{z}\rangle \\ \langle -\bar{z}|\hat{A}|z\rangle & \langle -\bar{z}|\hat{A}|-\bar{z}\rangle \end{pmatrix} |\psi\rangle_z = |\psi\rangle_z$$

call this the matrix representation
of \hat{A} in $|\pm z\rangle$ basis

$$\underbrace{\hat{A}}_{z} |\psi\rangle_z = |\psi\rangle_z$$

or
just

$$\hat{A}_z |\psi\rangle_z = |\psi\rangle_z$$

where $\hat{A}_z = \underbrace{\hat{A}}_{z} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} \langle z|\hat{A}|z\rangle & \langle z|\hat{A}|-\bar{z}\rangle \\ \langle -\bar{z}|\hat{A}|z\rangle & \langle -\bar{z}|\hat{A}|-\bar{z}\rangle \end{pmatrix}$

or $A_{ij} = \langle i|\hat{A}|j\rangle = \text{matrix element.}$

A little bit of hand waving physical interpretation:

$$\hat{A}|\psi\rangle = |u\rangle \quad |\psi\rangle = (\text{proj of } |u\rangle \text{ on } |z\rangle) |z\rangle + (\text{proj of } |u\rangle \text{ on } |-z\rangle) |-z\rangle$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \langle z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle z|u\rangle \\ \langle -z|u\rangle \end{pmatrix}$$

and same for $|u\rangle$

where

$$A_{ij} = \langle i|\hat{A}|j\rangle$$

well

$$\hat{A}|i\rangle \Rightarrow \underbrace{M}_{\substack{\text{rotation} \\ \text{analogy}}} \vec{r} = \vec{R}$$

so

\hat{A} rotates $|i\rangle$ to something new,
say

$$\hat{A}|i\rangle = |i'\rangle$$

then

$$A_{ij} = \langle i|\hat{A}|j\rangle = \langle i|i'\rangle$$

= proj of new
vector $|i'\rangle$

(So, for example)
onto basis $|i\rangle$

$$\begin{pmatrix} \langle z|\hat{A}|z\rangle \\ \langle z|\hat{A}|-z\rangle \\ \langle -z|\hat{A}|z\rangle \\ \langle -z|\hat{A}|-z\rangle \end{pmatrix} \begin{pmatrix} \text{proj of orig state } |u\rangle \text{ on } |z\rangle \\ \text{proj of orig state } |u\rangle \text{ on } |-z\rangle \end{pmatrix} = \begin{pmatrix} \text{new proj of } |u\rangle \text{ on } |z\rangle \\ \text{new proj of } |u\rangle \text{ on } |-z\rangle \end{pmatrix}$$

how \hat{A} changes $|z\rangle$
and then how much
of that projects onto z

012 ... enough... let's do an example!

Proj of spin angular momentum \hat{S}_z , $\frac{1}{2}$.

let's say, use, $|\pm z\rangle$ basis = complete!

Look See

$$\hat{S}_z |\pm z\rangle = \pm \frac{\hbar}{2} |\pm z\rangle$$

OK; given our complete $|\pm z\rangle$ basis

$$\hat{S}_z = \begin{pmatrix} \langle z | \hat{S}_z | z \rangle & \langle z | \hat{S}_z | -z \rangle \\ \langle -z | \hat{S}_z | z \rangle & \langle -z | \hat{S}_z | -z \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle z | \frac{\hbar}{2} | z \rangle & \langle z | \left(-\frac{\hbar}{2}\right) | -z \rangle \\ \langle -z | \frac{\hbar}{2} | z \rangle & \langle -z | \left(-\frac{\hbar}{2}\right) | -z \rangle \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\hbar}{2} \langle z | z \rangle & -\frac{\hbar}{2} \langle z | -z \rangle \\ \frac{\hbar}{2} \langle -z | z \rangle & -\frac{\hbar}{2} \langle -z | -z \rangle \end{pmatrix}$$

$$\hat{S}_z = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let's see if our scheme worked!

$$\hat{S}_z |+z\rangle \stackrel{?}{=} -\frac{\hbar}{2} |+z\rangle$$

in matrix formalism - - -

$|+z\rangle$:

$$\frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{cases} (1)(1) + (0)(0) \\ (0)(1) + (-1)(0) \end{cases}$$
$$= \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or

$$\hat{S}_z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{S}_z |z\rangle = \frac{\hbar}{2} |z\rangle$$

Then $|+z\rangle$:

$$\hat{S}_z |+z\rangle = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2} |+z\rangle$$

So

$$\hat{S}_z |\psi\rangle = |e\rangle$$

is true Always as long as

express $|\psi\rangle$ in $| \pm z \rangle$ basis

w/

\hat{S}_z = matrix elements of $\langle i | S_z | i \rangle$
over all the
(complete basis).

we see more!

For example ...

$$\hat{S}_z |x\rangle = ?$$

$$\hat{S}_z |y\rangle = ?$$

Trick... is \hat{S}_z is in z -basis, you'll need

$|x\rangle \notin |y\rangle$ in the $| \pm z \rangle$
basis.



Alternatively: You could perform a unitary
similarity transformation on \hat{S}_z to get it

in a \hat{S}_x basis

$$\hat{S}_z \rightarrow \hat{S}_x$$

med U.S.T., not bad at all, just
relation between
2 given sets of bases.

Unitary ... preserve
probability.

Big picture

$\hat{S}_z \xrightarrow{\text{U.S.T.}} \hat{S}_x$ give the equiv version
of \hat{S}_z but using \hat{S}_x
basis
so easy to do

$$\hat{S}_x |x\rangle = ?$$

See Townsend pg 48

$$\hat{A}_{x\text{-basis}} = \sum_{z\text{-basis}} A^{\text{(adjoint)}}_z$$

$$\text{where } S = \begin{pmatrix} \langle z|x \rangle & \langle z|z \rangle \\ \langle z|x \rangle & \langle z|z \rangle \end{pmatrix}$$

= matrix formed by finding the
projection's of the orig basis $|+z\rangle$
onto the new basis $|+x\rangle$