Currents, I, $\vec{J}$ & Continuity

Gauge Theories!

E.F. Deveny / BSC Physics PH358 Griffiths Ch. pt 5

Note Title 1/15/2005

I & $\vec{J}$ current density & continuity Equation

NOTE

Special Preview: pg 522: How $\vec{B}$ fields = relativistic effect!

in $\longrightarrow +\lambda, V_+$

$\longleftarrow -\lambda, V_-$

S = stationary if $V_+ = V_-$, Then for any pt in S

$\longleftarrow -\lambda, V_-$ $\quad$ $+\lambda, V_+ \longrightarrow$ } Both red. shifted equally so

idea: is there a frame where $V_+ \neq V_-$ then while neutral in S, it will not be in the other frame

in S $\longrightarrow +\lambda, V_+$ / $-\lambda, V_-$ $\Rightarrow$ looks neutral!

But carries Current $= (-\lambda)(V_-) + (+\lambda, V_+) = +2\lambda V_+$

$= I$

Consider

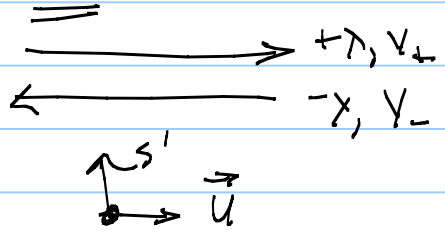
I

$q \xrightarrow{u}$

From S' traveling on q @ u

Then q is stationary in S'

But now



The V_+ & V_-'s are different

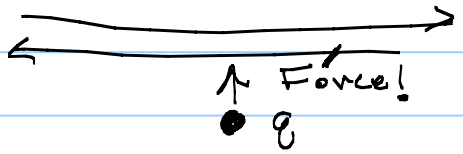
V_+ redshifted

V_- blue shifted

so λ^- is contracted } so higher density of − / Length

∴ q in S' sees less λ^+ & more λ^-

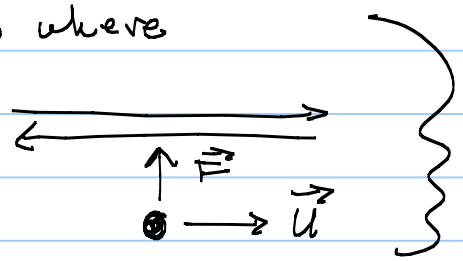
∴ q feels net (−) charge, while stationary in S'



can show using SR pg 522, $F' = \frac{-\lambda u}{\pi\varepsilon_0 c^2 s} \frac{qu}{\sqrt{1-(\frac{u}{c})^2}}$

Now

Back in S where

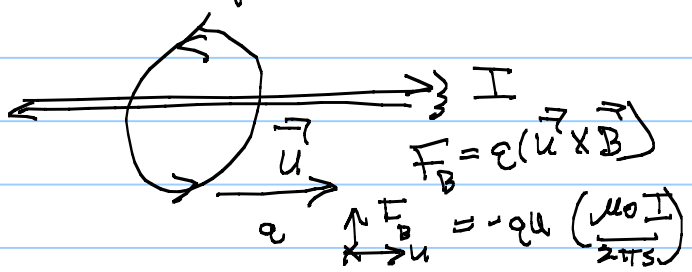


you now see Force that is ⊥ to $\vec{u}$

Wow!

Now if do the proper transform of F' back to S you get

$F = -qu\left(\frac{\mu_0 I}{2\pi s}\right) =$



So: just as relativity "mixed" space time

→ Spacetime 4-vector

$x^\mu = (ct, x_1, x_2, x_3)$

relativity also mixes E & M Field into

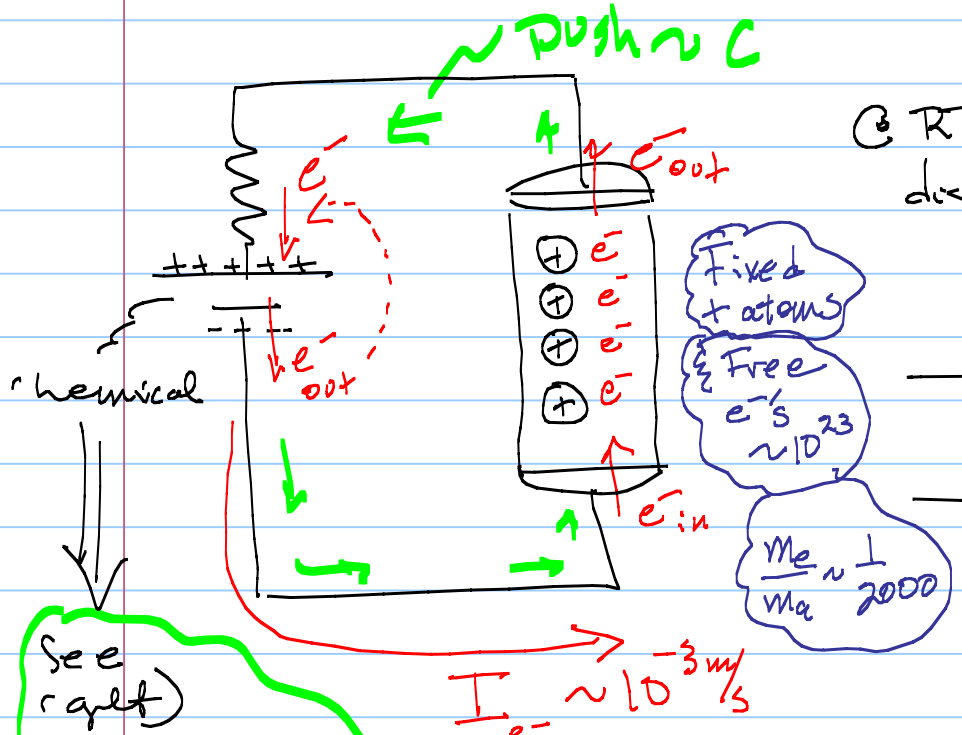
$F^{\mu\nu}$ = Field tensor

pg 536 Griffiths

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

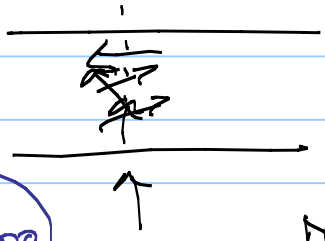
NOW BACK on Track: I & $\vec{J}$

So need to talk about currents & circuits



@ RT, maxwell Boltz distrib ⇒'s

$\bar{V} \cong 10^6$ m/s, 1% C



current = $\frac{\text{Net \# of } e^-\text{'s passing this pt}}{\text{time}}$

Normally = 0

However, add, potential e⁻'s are attracted to + thru "Free vide" of conductor so start "pushing" way toward + Thus |speed| ~ 10^{-3} m/s

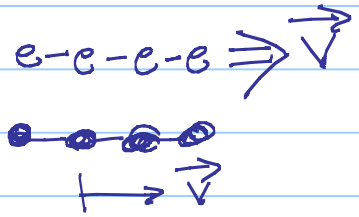
See (ralf) AJP, 51, 829 (1983) by D. Roberts on how real batteries work

* The push = |E| Force, moves @ speed of Light!
* Conductor stays Neutral thru out! (at least ≅ ly so, takes maybe $\frac{1}{c}$ = sec to reach it, may have net charge? ⇐ I dont know my idea! interesting?

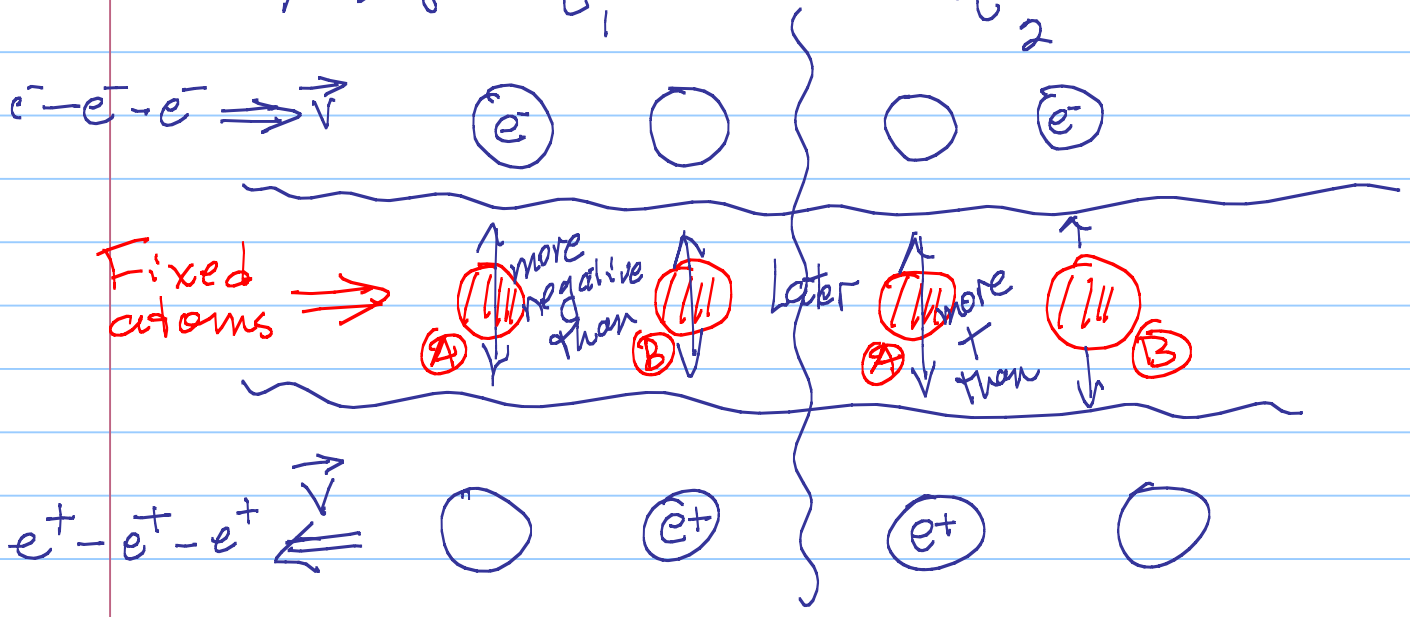
This all, ie idea of e^-'s pushing each other along, that

current looks like string of charge

$e^- - e^- - e^- \Rightarrow \vec{V}$



now, say t_1 t_2



Can see that to Fixed atoms A & B, makes no difference, if you are talking about

$e^- - e^- - e^- \Rightarrow \vec{V}$

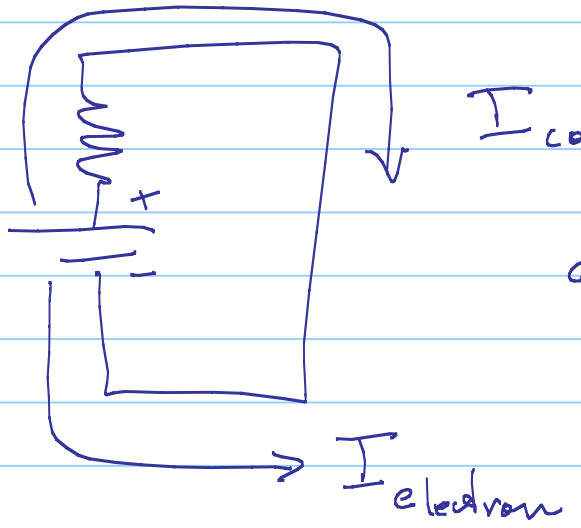
or $e^+ - e^+ - e^+ \Leftarrow \vec{V}$

Most physical electrical phenomena depend on $q\vec{V}$ so $-q\vec{V}_{\rightarrow} = -|qV|$

$+q V_{\leftarrow} = -|qV|$

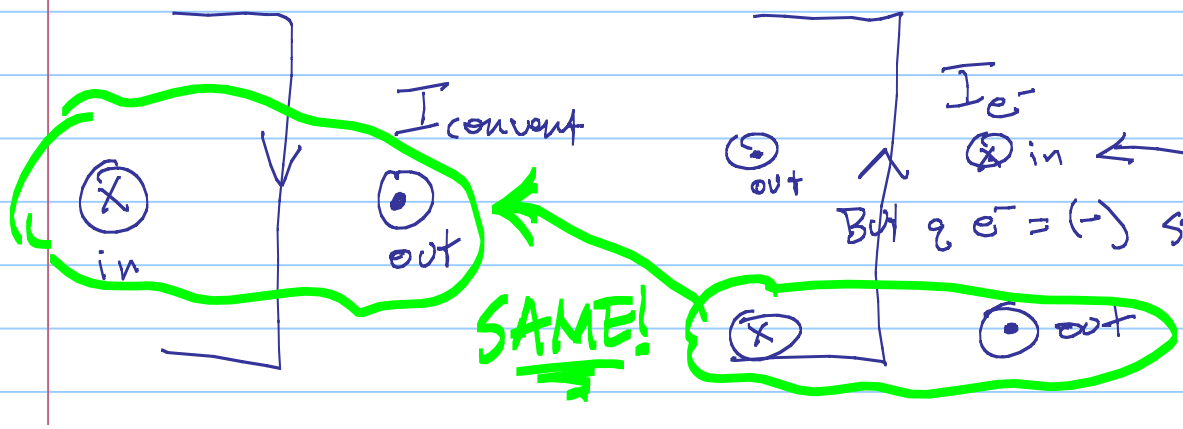
So, in fact, Ben Franklin didn't understand, well I know, that

$M_{atom} \cong 1836\ M_e$ so atoms = Fixed
e^- move

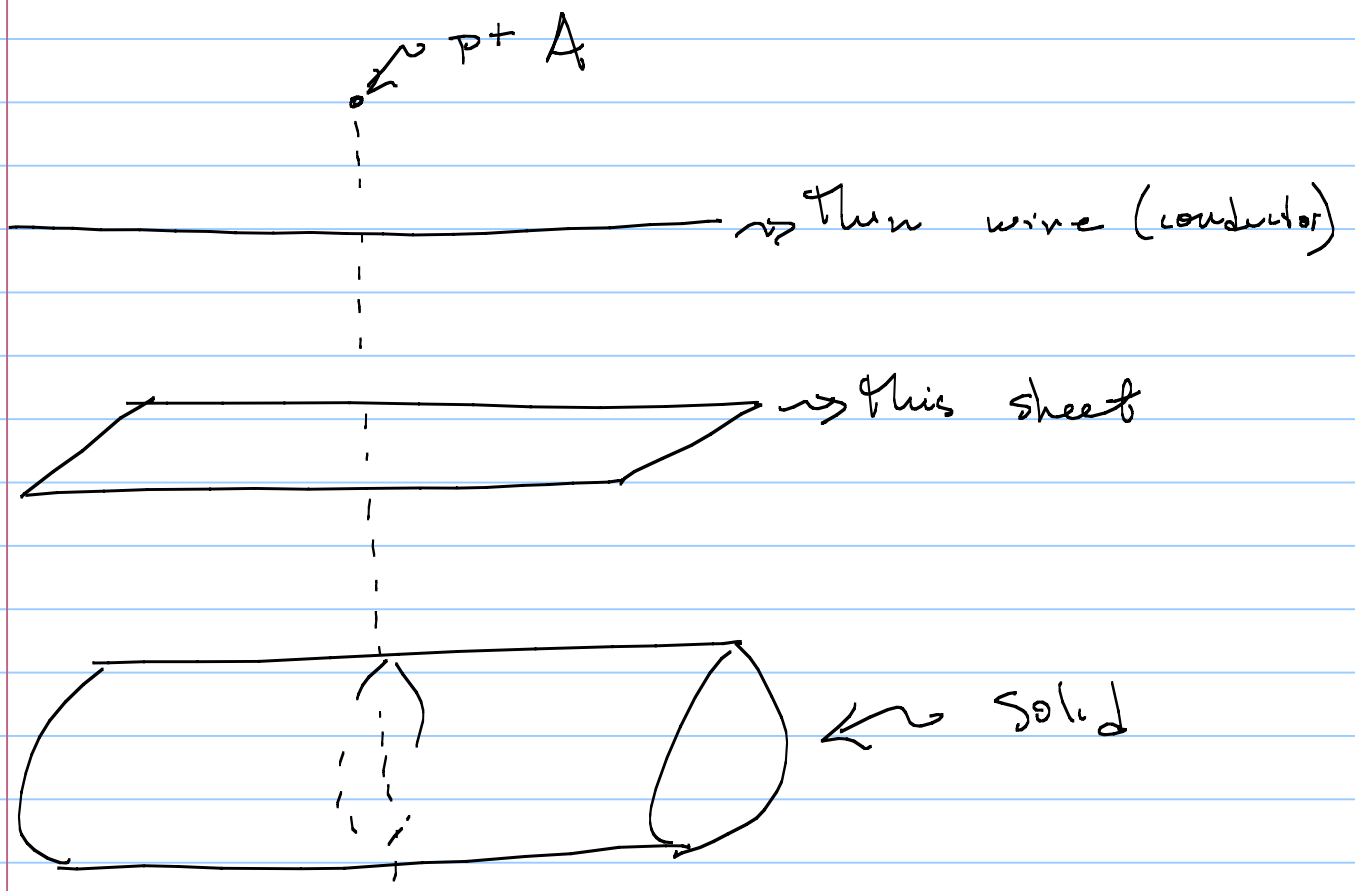


again plasma $\propto q\vec{v}$
q makes no difference
(except: Hall effect)

So even when we calculate $\vec{B}$ field due to I

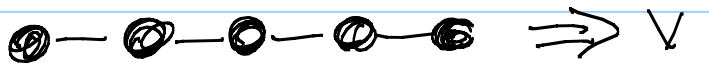


So current is $\equiv$ $\frac{\text{Charge}}{\text{time}}$ passing a given pt



(Clearly for thin wire, as we have discussed)

$e^- - e^- - e^- - e^-$ = e^-'s pushing each other



Looks a moving line of charge density, λ. Now 1.) because the density will dictate how many pass A

2.) which will depend on how fast that string of density is moving past A.

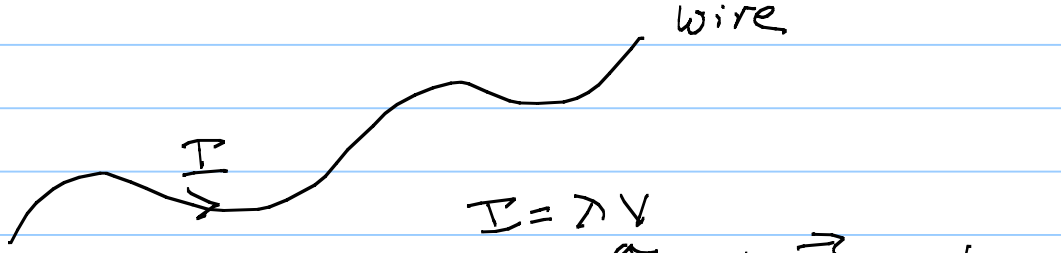
⇒ so that $I = \lambda V = \left(\frac{\text{coulombs}}{\text{m}}\right)\left(\frac{\text{m}}{\text{s}}\right) = \frac{C}{s}$

* note again

$\lambda = \frac{\xi}{m}$ so $I \propto \underbrace{\xi V}$

ξ it doesn't matter if it is $-\xi \rightarrow$ or $+\xi \leftarrow$

so for thin wire w/ linear charge dens. dy λ, we have



$I = \lambda V$

↳ note $\vec{V}$ = vector

so $\vec{I} = \lambda \vec{V}$ (current follows dir of $\vec{V}$ of λ which follows wire)

Now staying w/ wire for a moment

$$d\vec{F}_B = dq(\vec{v} \times \vec{B}) \quad \text{so}$$

$$\vec{F}_B = \int dq(\vec{v} \times \vec{B})$$

$$|dq| = \lambda |dl|$$

$$= \int (\vec{v} \times \vec{B}) \lambda \, dl$$

$$\vec{F}_B = \int (\vec{I} \times \vec{B}) dl$$

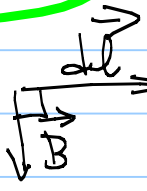
Now, usually $\vec{I} \cong$ constant

and recall dir of $\vec{I}$ must be same as wire, ie $d\vec{l}$

so: $\vec{F}_B = I \int (d\vec{l} \times \vec{B})$

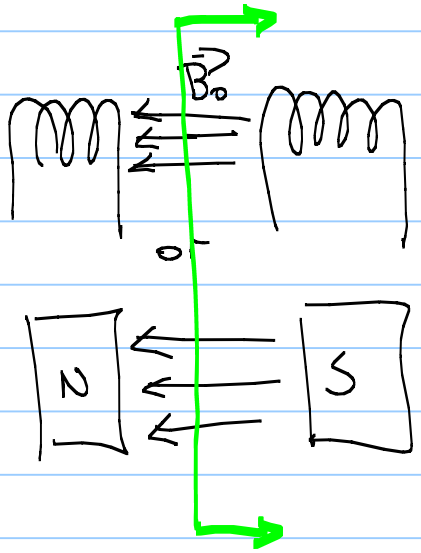
This is TRUE anywhere, $d\vec{l}$, on a wire w/ constant I in a known $\vec{B}$ field

* remember RtHR



$|dl||B|\sin\theta$ Then RtHR

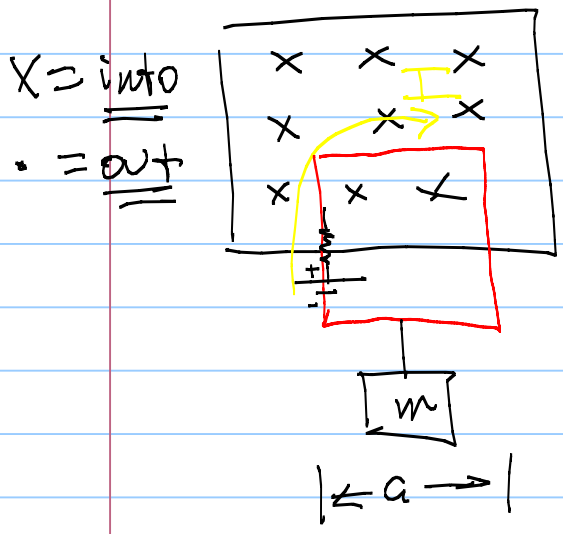
ex: 5.3 Griffith ⇐ also play games w/ $\vec{B}$ doing no work & kinds go way out!



Helmholtz coil

run current, I, thru loop shown ...

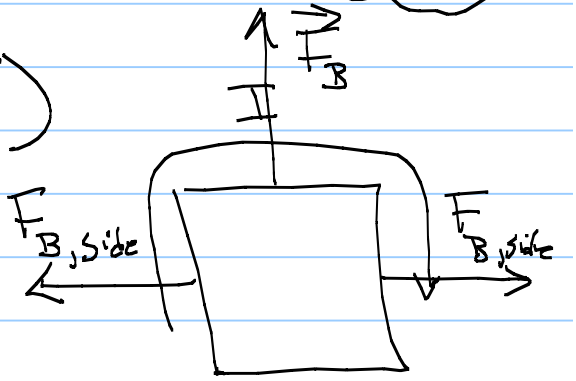
can the $\vec{B}$ field & I support m?



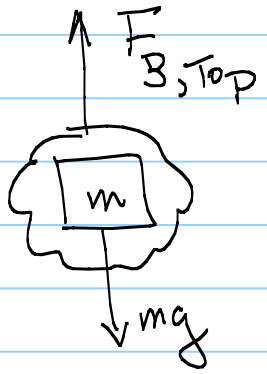
get idea already what to supply! Battery is doing work

Well $\vec{F}_{B\ on\ wire} = I \int (d\vec{\ell} \times \vec{B})$

clearly



So



Static Equi:

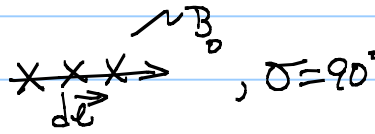
$$\Sigma \vec{F} = m\vec{a}$$

$$+\uparrow \Sigma F_y = ma_y$$

$$F_{B,Top} - mg = 0$$

$$F_B = mg$$

$$\rightarrow = I \int (d\vec{\ell} \times \vec{B}_0) = I \int_{Top} |d\ell|\,|B_0| \sin\theta$$



$$= I \int B_0 \, d\ell = I B_0 \int_{Top} d\ell = I B_0 a$$

(I constant)

OR since $I = \lambda V = \lambda \omega$ → (ω = charge, slow)

$\therefore$

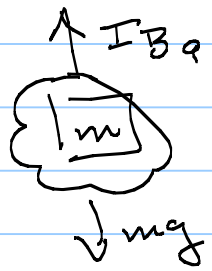
$$I B_0 a = mg$$

$$\& \; I = \frac{mg}{B_0 a}$$

weight is SUSPENDED in AIR

Now things get interesting if you ↑ the current, well

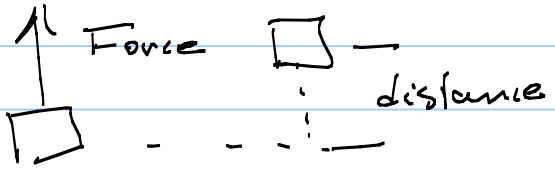
$F_B = IBa$ so



$IBa > mg$

& m will LIFT!

But wait.....



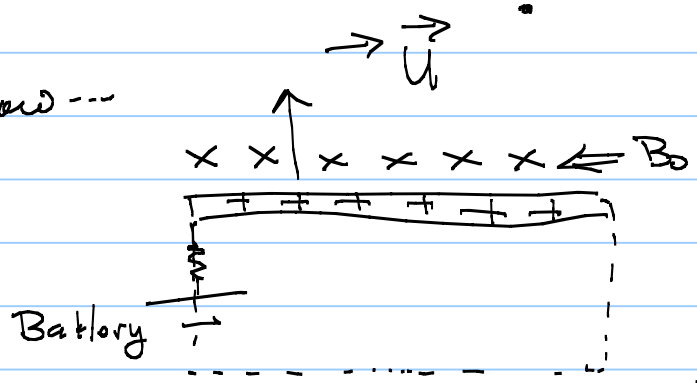
$W = \int \vec{F} \cdot d\vec{l}$ = WORK done HERE!

But wait..... who did the WORK?

$\vec{B}$ fields do not do work we thought!

Indeed, it turns out Battery is supply energy to do the WORK!

How...

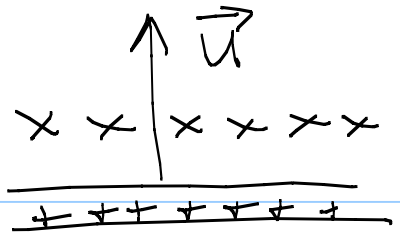


* recall, pretending

$e^- e^- e^- \leftarrow$

$e^+ e^+ e^+ \rightarrow$

these charges now have velocity ↑



induces a Force, $F_B = q\vec{v} \times \vec{B}$

$\uparrow \times = \leftarrow F_B$

$= IBq$

$F_B = (\lambda v) Bq$

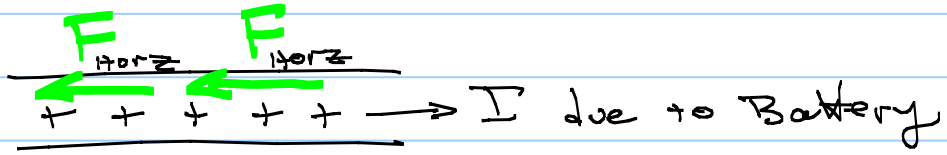
F_B horizontal

So

in order

For the battery to

maintain the current, I, thru the loop



against the Horizontal Force

The

battery must do work = opposite of $\int \vec{E} \cdot d\vec{\ell}$ induced

$W_{Batt} = \int \vec{F} \cdot d\vec{\ell}$

$= \int \lambda v B a (\leftarrow) \cdot d\vec{\ell}$

$d\vec{\ell}$ in dt

$= \lambda v B a \int (\leftarrow) \cdot (\rightarrow) w\, dt$

$d\vec{\ell} = w\, dt\, (\rightarrow)$

where $I = \lambda w$

$= \lambda w B a \int v\, dt$

$W_{Bat} = IBah$; h = height lifted

So, the battery had to do this work

which is exactly, incidently, equal to the work you might have ascribed to the B field (incorrectly) from the begging sing

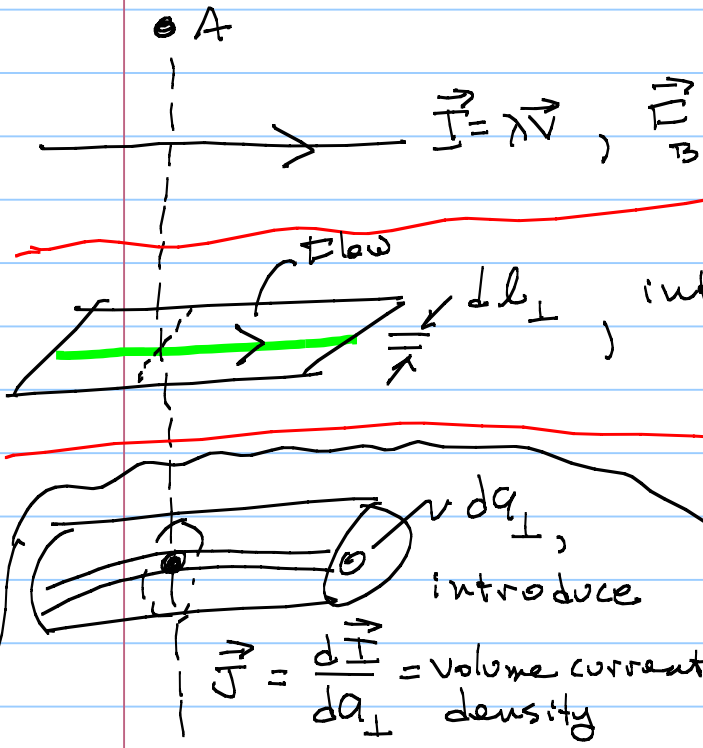
$$F_{B \text{ on wire}} = IB\ell$$

and it lifted, h

$$= IBah.$$

Moral: $\vec{B}$'s don't do work & hope to be closer to "see" what is really going on.

So current = $\frac{\text{How much charge}}{\text{time}}$ past a pt



$\vec{I} = \lambda \vec{v}$, $\vec{F}_B = I \int (d\vec{\ell} \times \vec{B})$

, introduce $\vec{K} = \frac{d\vec{I}}{dl_\perp}$ = surface current density

$\vec{K}$ = $\frac{\text{current}}{\text{unit width}}$ on a surface

introduce

$\vec{J} = \frac{d\vec{I}}{dq_\perp}$ = volume current density

= $\frac{\text{current}}{\text{Area}\perp \text{ to flow}}$ in a volume

well as I = e-e-e = moving string of charge = $\lambda \vec{v}$

So just as in other case's

$\vec{J} = \rho \vec{v} = \left(\frac{\text{charge}}{\text{volume}}\right) \vec{v}$

moving

$= \frac{C}{m^3} \frac{m}{s} = \left(\frac{C}{s}\right) \frac{1}{m^2} = \frac{I}{m^2}$

Clearly $\vec{F}_B = \int dq (\vec{v} \times \vec{B})$

$= \int_{Vol} (\vec{v} \times \vec{B}) \rho \, dV = \int_{Volume} (\vec{J} \times \vec{B}) \, dV$

Current on surface looks like

$\vec{K} = \sigma \vec{v} = \left(\frac{\text{charge}}{\text{Area}}\right) \vec{v}$

moving

$= \frac{C}{m^2} \cdot \frac{m}{s} = \left(\frac{C}{s}\right) \frac{1}{m} = \frac{I}{\text{length}}$.

Clearly $\vec{I}_B = \int d\vec{F}_B = \int dq (\vec{v} \times \vec{B})$

$= \int_A (\vec{v} \times \vec{B}) \sigma \, dA = \int_A (\vec{K} \times \vec{B}) \, dA$

Now: The continuity Equation which is HUGE throughout physics ⇒ Related to CAUSALITY and all field theories!

recall idea of Flux!

$$\text{Flux} = \int_A \vec{V} \cdot d\vec{A} = \text{How much of vector field } \vec{V} \text{ flows out of } \vec{A}$$

so

$$\oint \vec{E} \cdot d\vec{A} = \text{How much of } \vec{E} \text{ flows out of } A$$

Then

$$\underbrace{\oint \vec{E} \cdot d\vec{A}}_{\text{closed around } \forall} = \underbrace{\int_{\forall} \vec{\nabla} \cdot \vec{E}\, d\forall}_{\text{net amount of } \vec{E} \text{ out}} = \frac{Q_{\text{free}}}{\varepsilon_0}$$

$$\frac{\text{sources of } \vec{E}}{\text{volume}} = \frac{\text{Charge}}{\text{volume}} \left(\frac{1}{\varepsilon_0}\right)$$

ie charges "source" $\vec{E}$

So: Let's ask what "sources" current density $\vec{J}$ vector field!

Just write the same equations for vector field $\vec{E}$ (True for all vector fields)

$$\oint \vec{J}\cdot d\vec{A} = \int_{\forall} (\vec{\nabla}\cdot\vec{J})\, d\forall = \frac{\text{Sources of } J}{\text{volume}}$$

close area

Net Flux of J out of A

$\frac{\text{source of } J}{\text{volume}}$

work w/ this again cause anticipate divergence problems in E&M as $r \to 0$ (recall invoked Dirac delta function to mimic TRUE pt source charges which do not seem physical

well

$$\int \vec{J}\cdot d\vec{A} = \int J\, dA_{\perp \text{ to } J} = \int \frac{I}{(m^2)}(m^2) = I$$

so

$$\oint \vec{J}\cdot d\vec{A} = \int_{\forall} (\vec{\nabla}\cdot\vec{J})\, d\forall$$

Net amount of current out = Net current out where $\vec{\nabla}\cdot\vec{J} = \frac{\text{current}}{\text{volume}}$

$$= \ominus \frac{d}{dt}\int_{\forall} \rho\, d\forall' = -\int \left(\frac{\partial \rho}{\partial t}\right) d\forall$$

if its charge & leaving must be @ expense, loss of ρ inside

So can conclude

$$\int_{\forall} (\vec{\nabla} \cdot \vec{J})\, d\forall' = -\int \left(\frac{\partial \rho}{\partial t}\right) d\forall'$$

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

Read: $-\frac{\partial \rho}{\partial t}$ "sources" divergent $\vec{J}$ (current densities)

or 1.) time rate of change of charge
2.) Sources a current density that diverges out & away!

Now this is HUGE is Q.F.T.

Because, special relativity is combined, which allows particles to be created & destroyed!

So if you have

t_1 t_2

Then

e^- nothing

It is OK as long as a current was "Produced"

Now this current is a "field"
& essentially CAPTURES conservation law!

OK to lose a particle, as long as a field was produced, that "carries" away all the properties of the particle!

So idea of Conservation } To Conserve a quantity ⇒ introduce a field

& idea of local vs non local effects.

↳ can you lose a particle, need to make a field, but that field has properties notably that it cannot travel faster than c

So Universe A

$e^- + e^+ \rightarrow$ nothing

No

field (γ) must be created that carries conservation & travels @ c → Then Con

Universe B

? pop up instantly $e^- + e^+$

The basic idea in addition to conservation is:

Conservation $\Rightarrow$'s Field

Then Global conservation, say Charge is Conserved in the Universe

to make it local

means

need the Field to carry it & the Field has properties!

see: R. Mills AJP 1980 "Gauge Theory"

So: Idea: Q.M. says world invariant to absolute phase,

$\psi' \rightarrow e^{i\phi}\psi$ so $|\psi^*\psi| = |\psi^*\psi|$

Thus

is global in sense $e^{i\phi}$, $\phi \neq S(\vec{x},t)$ = everywhere all @ once.

To make it local need

$\phi = S(\vec{x},t)$

Now to make physics The same globally & locally need The Lagrangians to match

$$\mathcal{L} \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varepsilon}}\right) - \frac{\partial \mathcal{L}}{\partial \varepsilon} = 0$$

So

Global vs Local

$\phi \neq S(\varepsilon, t)$ not effected

$\phi = S(\varepsilon, t)$ effected by the space & time derivs

So to make equal

need to ADD

terms that cancels

Then new deriv.

Now in Q.F.T. $|\psi| =$ particle fields

So Terms added = Fields

& in this CASE, the fields HAVE ALL the same properties of Maxwell's equations

Thus E & M Fields!

So: to keep the universe invariant to local changes in phase

The WORLD must have $\vec{E}$ & $\vec{M}$ fields!

This is a guage theory:

others:

1.) Color invariance: $\begin{pmatrix} u_r \\ u_b u_g \end{pmatrix} \rightarrow \begin{pmatrix} u_g \\ u_b u_r \end{pmatrix}$ ⇒'s world must have all the gluons ⇒ thus QCD

world can't tell difference globally so to make it true locally need to add & need fields

2.) Isospin ⇒'s

$n \rightarrow p$ (except charge)

⇒'s need 4 fields (add, introduce)

massless fields @ C

But 2 carry charges

γ
γ^-
γ^+
γ^0 } interact w/ Higgs fields & get heavy!

⟹ E & M

W^-
W^+
Z^0 } weak ≈ nuclear by neutral & charged "currents"

Brings us Back to start!