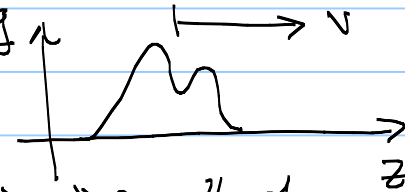
as $S(z,t) = g(z - vt) =$

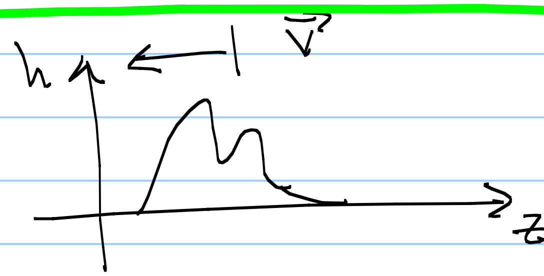


traveling "wave" of fixed shape

is a soln to

$$\boxed{\frac{\partial^2 S}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 S}{\partial t^2}} \Leftarrow \text{wave equat}$$

$S(z,t) = h(z + vt) =$



is a traveling wave to the $\leftarrow$
propagating in the neg z dir

Can show by substitution that the wave equation is linear so the most gen soln is

$$S(z,t) = g(z - vt) + h(z + vt)$$
$$\longrightarrow \; + \; \longleftarrow$$

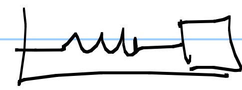
ie: The sum of 2 solutions is itself a soln.

EVERY soln to the wave equat can be expressed as

Like the Simple Harm motion

$$\sum \vec{F} = m\ddot{x}$$

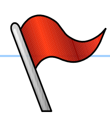
$$-kx = m\ddot{x}$$

= 

The Wave equation

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Both are ubiquitous in physics!

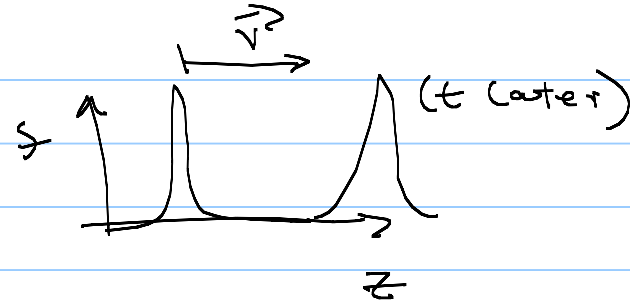


H.W: Griffs 9.1 $\Rightarrow$ f_1, f_2, f_3 & f_4 satisfy W.E.

& 9.2 $\Rightarrow$ show standing wave = $\rightarrow$ + $\leftarrow$

Now waves such as

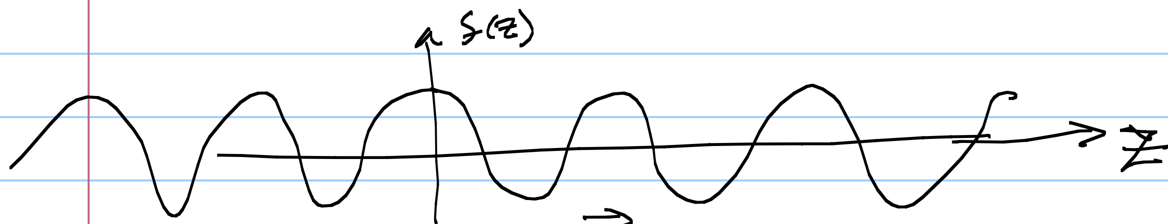
$\xi_1(z,t) = A e^{-b(z-vt)^2} =$



But clearly, the sinusoidal wave

$\xi(z,t) = A\cos[k(z-vt)+\delta]$ also a soln to the W.E. & is traveling wave →

looks like



& travels

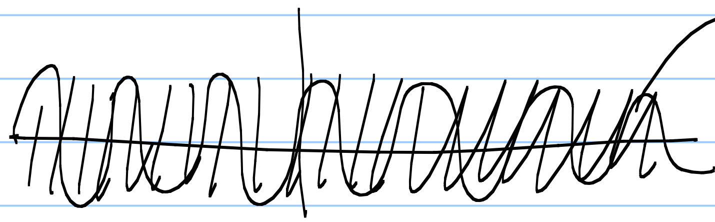
Now this will be hugely significant W.E. soln
Because as we know Sin & Cos of different frequencies

1.) either denumerably ∞ $\sum_{n=0}^{\infty}$

or

2.) continuously ∞ $\int_{-\infty}^{+\infty} [\ ]\, df$ ↳ # freq.

are linearly independ &
form an ∞ basis set in function space



ie the Σ
or $\int$
completely fill that

space so any function can built as a linear
combo of them!

So lets study prop's of sinusoidal wave

$$f(z,t) = A\cos[\underbrace{k(z-vt)}_{\text{traveling wave}} + \delta]$$

as general as possible (ie plays no role in W.E. derivatives but used to meet the B.C.'s along w/ A

$$\underbrace{[k(z-vt)}_{\text{argument}} + \delta] = \text{radians}$$

δ: phase (radians)

$$k(\underbrace{z-vt}_{\text{distance}}) = \text{rad}$$

$$k = \frac{\text{rad}}{\text{dis}}$$

$$kz - kvt$$

$$\underbrace{\frac{\text{rad}}{\cancel{m}}\frac{\cancel{m}}{s}}\, t$$

$$\omega t$$

so $\omega = kv = \dfrac{2\pi}{T}$

where

T = 1 period of time

in T, in z

distance

1λ

Plot w/ excel & scroll bar!

TPT article

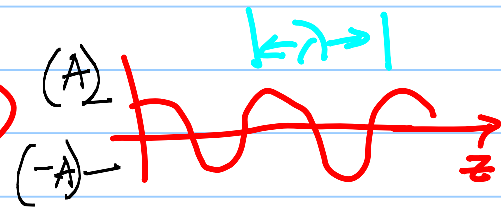
$T = \frac{1\lambda}{v}$

So $\omega = kv = \frac{2\pi}{T} = \frac{2\pi}{\frac{\lambda}{v}}$

So $k = \frac{2\pi}{\frac{\cancel{v}\lambda}{\cancel{v}}} = \frac{2\pi}{\lambda}$

$$f(z,t) = A\cos\left[kz - \omega t + \delta\right]$$

governs shape of wave as $f(z)$ (@ fixed t)



& $k = \frac{2\pi}{\lambda} = \frac{\text{rad}}{\text{m}} \equiv$ Wave # but it is like wave 'space' radial velocity ..... $\frac{\text{radians}}{\text{meter}}$

governs @ fixed z

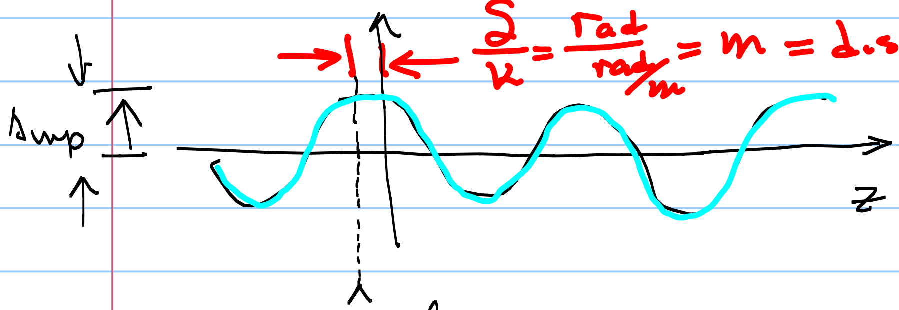
T

$\omega = \frac{2\pi}{T} = \frac{\text{rad}}{\text{sec}} =$ angular velocity

@ any pt.

So k = space radial velocity, then $\frac{\delta}{k} = \frac{\text{rad}}{\frac{\text{rad}}{\text{m}}} = \text{m}$

So δ/k = distance by which entire wave is delayed .... i.e. @ $\delta(z, t=0)$



$\frac{\delta}{k} = \frac{\text{rad}}{\text{rad}/m} = m$ = distance wave started "before" central peak @ $z=0$ & $t=0$

is @ cos (0)

now $\cos[kz - \omega t + \delta]$, with $t \to 0$

@ $z = 0$

cos is already $\cos[\delta]$

so as is wave started earlier

So now let time go →
and the max come thru @ $z = 0$ some time later

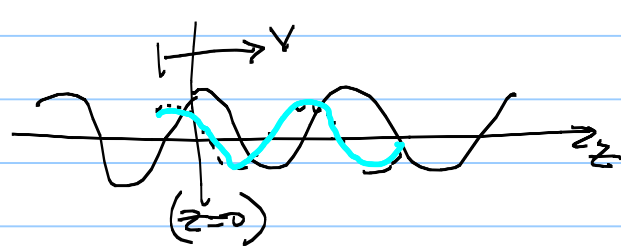
$$\cos[0 - \omega t + \delta]$$

@ $\omega t = \delta$

$$t = \frac{\delta}{\omega} = \frac{\text{rad}}{\frac{\text{rad}}{\text{sec}}} = \text{sec}$$

The central peak passes $z = 0$

thus a neg δ's ⇒ the central peak is advanced ahead of $z=0$ @ $t=0$



So

$$f(z,t) = A\cos(kz - \omega t + \delta) = \text{delayed sinusoidal wave} \longrightarrow$$

& $$f(z,t) = A\cos(kz + \omega t - \delta) = \underbrace{\text{delayed}}_{\text{delay is: comes early from right.}} \text{ sinusoidal wave} \longleftarrow$$

Now we have to have a complex algebra break.

Say you have $s_1 = \cos(.7\pi + x)$

$s_2 = \cos(.7\pi + y)$

$s_3 = s_1 + s_2 = ?$ yikes you'd have to do lots of trig identities

Heres the idea:

elevate s's $\to \tilde{s}$ complex (you've added imaginary part But thats ok, its linearly indep)

so $s_1 \to \tilde{s}_1 = e^{i(.7\pi + x)}$

$s_2 \to \tilde{s}_2 = e^{i(.7\pi + y)}$

Then $\tilde{s}_1 = e^{ix} e^{i.7\pi}$

$\tilde{s}_2 = e^{iy} e^{i.7\pi}$

$\tilde{s}_3 = \tilde{s}_1 + \tilde{s}_2 = (e^{ix} + e^{iy}) e^{i.7\pi}$

$= \tilde{s} e^{i.7\pi}$

$|\tilde{s}|$

recall our 3 notation for 3 algebra's

1) Vector $\vec{A}\vec{B}$ → $\vec{A}\cdot\vec{B}$, $\vec{A}\times\vec{B}$

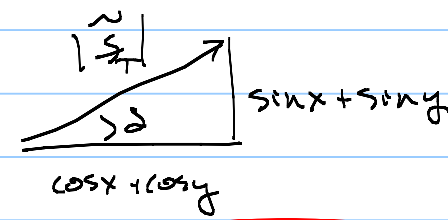
recall 1 & 2 = matrix algebra

2) Tensor $\vec{r}' = A\,\vec{r}$

3) complex $\tilde{C} = \tilde{A}\tilde{B}$

$$(\cos x + i\sin x) + \cos y + i\sin y$$

$$(\cos x + \cos y) + i(\sin x + \sin y)$$



$$\delta = \tan^{-1}\left(\frac{\sin x + \sin y}{\cos x + \cos y}\right)$$

$$S_T = |\tilde{S}_T| = \sqrt{(\cos x + \cos y)^2 + (\sin x + \sin y)^2} = \tilde{S}_T^{*}\tilde{S}_T$$

$$\tilde{S}_3 = \tilde{S}_1 + \tilde{S}_2 = \tilde{S}_T e^{i\,.7\pi}$$

$$= S_T e^{i\delta} e^{i\,.7\pi} = S_T e^{i(.7\pi+\delta)}$$

$$\tilde{S}_3 = \; = S_T\left(\cos(.7\pi+\delta) + i\sin(.7\pi+\delta)\right)$$

Now wanted $S_3 = S_1 + S_2$ so take real parts

$$S_3 = \mathrm{Re}(\tilde{S}_3) = S_T\cos(.7\pi+\delta)$$

Thus we will write our sinusoidal waves as

$$f(z,t) = A\cos(kz - \omega t + \delta)$$

$$f(z,t) = Re(\tilde{f}(z,t)) = Re\left[\tilde{A}e^{i(kz-\omega t+\delta)}\right]$$

or most conveniently

$$\tilde{f}(z,t) = \tilde{A}\,\underbrace{e^{i(kz-\omega t)}}_{\text{separates phase into}}$$

H.W. Griffiths 9.3

Now as $\Rightarrow$'ed sin & cos = ∞-D Basis set
And
any wave can be written as

$$\tilde{f}(z,t) = \int_{k=-\infty}^{k=+\infty} \tilde{A}(k)\, e^{i(kz-\omega t)}\, dk$$

$k = \frac{rad}{m}$

so A units of m

$f(z-vt) \rightarrow$ note $+k \Rightarrow e^{i(kz-\omega t)}$ = sin wave $\longrightarrow$

$f(z+vt) \rightarrow$ note $-k \Rightarrow e^{i(-kz-\omega t)} = e^{-i(kz+\omega t)} = \longleftarrow$

Looks alot like

periodic funct = $\sum_{n=0}^{\infty}\left[a_n \cos n\omega z + b_n \sin n\omega z\right]$

= Fourier Series

but this is actually Fourier Transform
which
is not restricted to periodic funct
but instead
can represent single pulses that travel as waves!

Note $\displaystyle\int_{-\infty}^{+\infty} \tilde{A}(k)\, e^{i(kz-\omega t)}\, dk$

that $k = \dfrac{2\pi}{\lambda}$ & $\lambda\nu = v$
so $\lambda = \dfrac{v}{\nu} = \dfrac{v}{f}$

$\omega =$ $= \left(\dfrac{2\pi}{v}\right)(f_{req})$

or

$\displaystyle\int_{-\infty}^{+\infty} A(f)\, e^{i\left(\frac{2\pi}{v} f - \omega t\right)}\, (\ )\, df$

So Need to have freq from $-\infty$ + $+\infty$ to recreate a single traveling wave, pules

So our convention is, for now, well established sinusoidal waves

$$f(z,t) = A\cos(kz - \omega t + \delta)$$

That we will elevate to complex $\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$

Then do all work w/ $\tilde{f}$

& @ the end, because want $\cos(\ )$

we will take $\mathrm{Re}[\text{final results}]$

* if want $\sin(\ )$ looking answers

start w/ $f(z,t) = A\sin(kz - \omega t + \delta)$

$\hookrightarrow \tilde{A}e^{i(kz-\omega t)}$

& use imaginary part of answer @ end of day.

Idea is that the complex algebra procedures keep the Re & Im parts linearly indep so in end, you have 2 solutions & can use either. But along way, much easier to work w/ $e^{i\alpha}$

So lets do Traveling wave problems

I.) Reflection & Transmission @ Boundaries!

Consider

thin string $\qquad$ thick string

$\mu_1 = \frac{m}{L}$ $\qquad$ $\mu_2 = m/L$

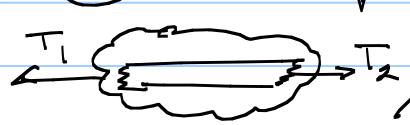
$$\mu_2 > \mu_1$$

& $$\frac{\partial^2 y}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

where T = Tension in rope

so $v \equiv$ wave velocity = different in each media! $= \sqrt{\frac{T}{\mu}}$

Now key to note T must be SAME throughout rope because @ all pt's



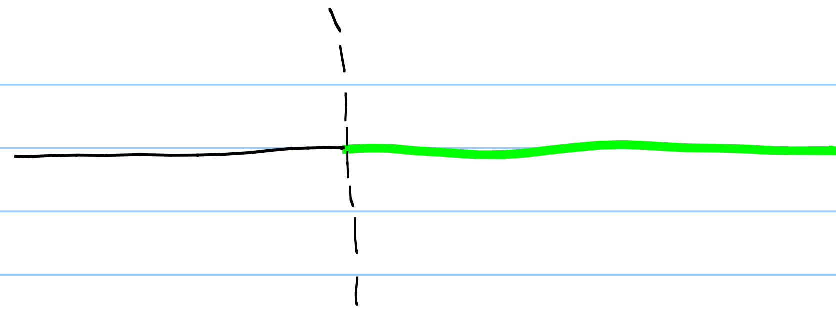
$\sum \vec{F}_{ext} = m\ddot{x}$

$0 = m\ddot{x}$

otherwise 'pieces' of rope would accel w/ respect to other pieces

$z=0$ for convenience

now see what happens to traveling wave $\longrightarrow$



$z<0$ | $z>0$

$$\tilde{f}_{Inc'd} = \tilde{A}_I e^{i(k_1 z - \omega t)} \Longleftarrow$$

we noted v is different in each

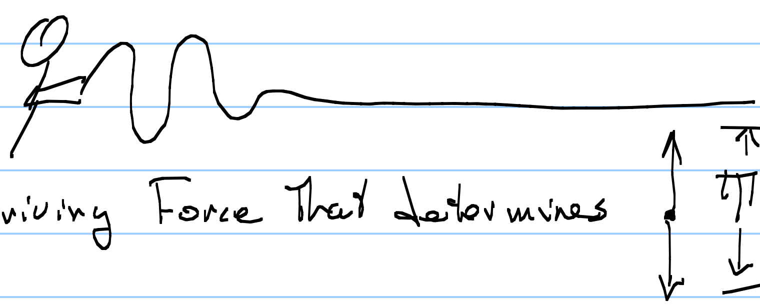
Since $k = \frac{2\pi}{\lambda} = \frac{2\pi}{vT}$

now v changes in μ_1 & μ_2

so $k_1 = \frac{2\pi}{v_1 T}$

But what about the period T?

well



Driving Force That determines $\updownarrow$ T up & down period.

So can imagin that the Amp may vary & that the speed $\longrightarrow$ may vary

But $\updownarrow$, T is defined by $\overset{\omega}{S}_I$

and is the same everywhere $Z<0$ & $Z>0$

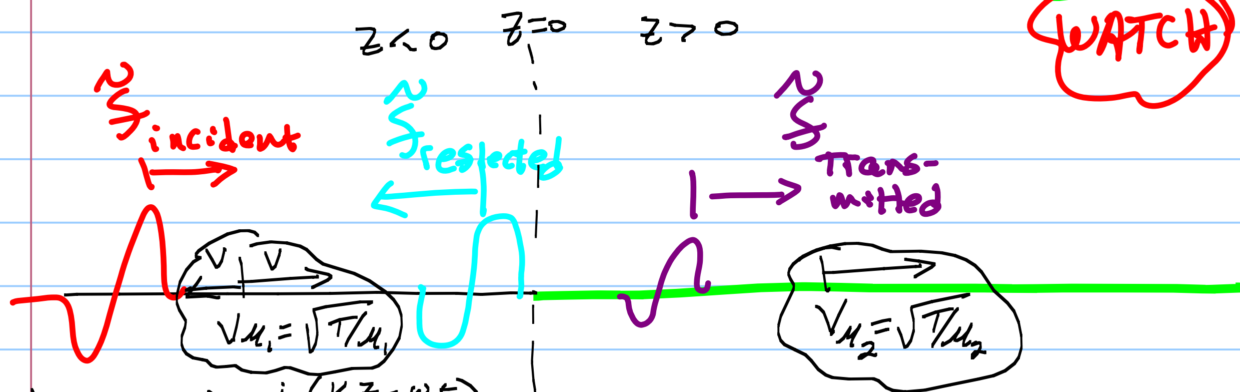
It's like a conservation of wiggles or frequency (Hz) even @ the interface!

$k_i = f(\mu_i)$

but ω = same everywhere!

WATCH

So we may have



$$\tilde{\xi}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)}$$

$$\tilde{\xi}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)}$$

$$\tilde{\xi}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)}$$

Then

$$\tilde{\xi}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)} & z<0 \\ \tilde{A}_T e^{i(k_2 z - \omega t)} & z>0 \end{cases}$$

Now @ $z=0$, $\xi(0-\varepsilon, t) = \xi(0+\varepsilon, t)$; $\varepsilon \to 0$

ie displacement either side $z=0$

$\varepsilon \to 0$ otherwise Break!

ie $\xi(z,t)$ must be continuous } Math

& The deriv of $\xi(z,t)$ must also be continuous

since $\frac{\partial \xi}{\partial z} \Rightarrow$ is veloc $\rightarrow$

Then @ the "knot" @ $z=0$

$$\left.\frac{\partial \xi}{\partial z}\right|_{\varepsilon+} = \left.\frac{\partial \xi}{\partial z}\right|_{\varepsilon-}$$

otherwise knot would move away from each other

Since sin & cos are linearly indep, these same conditions must apply to ξ

$$\tilde{\xi}(0^-,t) = \tilde{\xi}(0^+,t) \qquad \left.\frac{\partial \tilde{\xi}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{\xi}}{\partial z}\right|_{0^+}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T \qquad k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T$$

$$\tilde{A}_R = \tilde{A}_T - \tilde{A}_I \qquad \tilde{A}_T = \frac{k_1}{k_2}(\tilde{A}_I - \tilde{A}_R)$$

get in form of $\tilde{A}_I$

$$\tilde{A}_R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)\tilde{A}_I$$

$$\tilde{A}_T = \left(\frac{2k_1}{k_1 + k_2}\right)\tilde{A}_I$$

= outputs in terms of 'given' input

and $k = \frac{2\pi}{\lambda} = \frac{2\pi}{vT}$ so $k_i \propto \frac{1}{v_i}$

$\therefore$

$$\tilde{A}_R = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{A}_I$$

$$\tilde{A}_T = \left(\frac{2v_2}{v_2 + v_1}\right)\tilde{A}_I$$

OK now

or

$$A_R e^{i\delta_R} = \left(\frac{V_2 - V_1}{V_2 + V_1}\right) A_I e^{i\delta_I}$$

$$A_T e^{i\delta_T} = \left(\frac{2V_2}{V_2 + V_1}\right) A_I e^{i\delta_I}$$

if 2nd string is lighter $\mu_2 < \mu_1$

Then $v = \sqrt{\frac{T}{\mu}}$; $V_2 > V_1$

Then clearly $A_R = \left(\frac{V_2 - V_1}{V_2 + V_1}\right) A_I$ & $e^{i\delta_R} = e^{i\delta_I}$

or

$\delta_R = \delta_I$

& $A_T = \left(\frac{2V_2}{V_2 + V_1}\right) A_I$ & $e^{i\delta_T} = e^{i\delta_I}$

so $\delta_T = \delta_R = \delta_I$

Now: If μ_2 heavier, $\mu_2 > \mu_1$

& $V_2 < V_1$

Then $A_R e^{i\delta_R} = -1\left(\frac{V_1 - V_2}{V_2 + V_1}\right) A_I e^{i\delta_I} = \left(\frac{V_2 - V_1}{V_2 + V_1}\right) A e^{i(\delta_I + \pi)}$

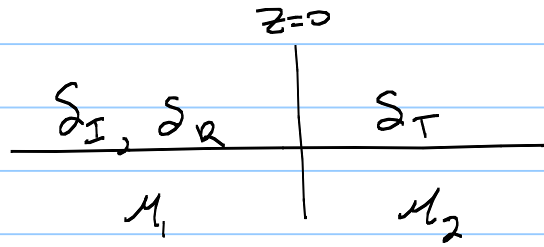
so

$$A_R = \left(\frac{V_2 - V_1}{V_2 + V_1}\right) A_I \text{ but } e^{i\delta_R} = e^{i(\delta_I + \pi)}$$

&

A_T = same!

now since $A_R = \left(\frac{v_1 - v_2}{v_2 + v_1}\right) A_I$ w/ $\delta_I = \delta_R + \pi = \delta_T$



$\delta_R = \delta_I - \pi$
or $\delta_T - \pi$

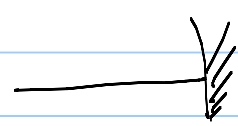
w/ respect to δ_I

$$\cos(-k_1 z - \omega t + \delta_I - \pi)$$

$$\cos(\alpha - \pi) = \cos\alpha\cos\pi + \cancel{\sin\alpha\sin\pi}$$

$$= -\cos(-k_1 z - \omega t + \delta_I)$$

So the reflected wave is "upside down" w/ respect to δ_I.

FURTHER: if  effectively $\mu_2 = \infty$ $\therefore$ $v_2 = \sqrt{\frac{T}{\mu}} = 0$

& $A_R = A_I$
& $A_T = 0$ $\Big\}$ entirely reflected!

But inverted!