E.F. Deveney / BSC Physics / PH458 E&M2

Magnetic Vect Pot
&
mag-dipole multipole
expansion

Note Title 1/25/2005

$\vec{F}_L = q\vec{E} + q\vec{V} \times \vec{B}$

- ↳ brute Force
- ↳ symm
- ↳ Electric Scalar potent, $V(r)$

$\vec{E}(\vec{r}) = -V(\vec{r})$

- → brute Force (B-S)
- → Symmetry (Amphere)
- → Magnetic Vector potential $\} = \vec{A}(\vec{r})$

$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$

Might have guessed
work towards an $\vec{A}$

The Bummer is, $\vec{A}$ is a vector so no huge
saving grace in problems.

However, as mentioned: Nature "follows"
Euler-Lagrange minimization of action
integral ↳ $S = \int \mathcal{L}\, dt$

when

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - qV + \frac{q}{c}\vec{A}\cdot\vec{\dot{x}}$$

so

$$E.L. = \underbrace{\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0}_{\text{essentially } \Sigma \vec{F} = m\vec{a}}$$

then E.L. reduces
to

$$\underbrace{q\vec{E} + q\vec{V} \times \vec{B}}_{F_{Lorentz}!} = m\vec{a}$$

(see Appendix A? in Townsend)

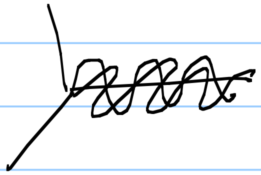
So see that the magnetic vector potential plays an integral role Fundamentally.

In fact when you quantize the 'real', transverse self-propagating, charge-free E&B field you get

$$\hat{H} \propto (E^2 + B^2)$$

you get both E & B in terms of $\vec{A}$

and then you can see that



= 'real' transvere harmonic oscillators we call photons!

See Chapter 14 Townsend.

& HUGE in Griffith Chapter 8 & on.

When you see 'real' self propagating E&B field think $\boxed{\vec{A}}$ so BIG!

OK: Vector Potential, $\vec{A}$

in electrostatics

$$\vec{\nabla}\times\vec{E}=0 \quad , \Rightarrow \quad \vec{E}=-\vec{\nabla}V$$

Now in magnestatics (& Dynamics!)

$$\vec{\nabla}\cdot\vec{B}=0$$

$$\Rightarrow\text{'s} \quad \vec{B}=\vec{\nabla}\times\vec{A}$$

Griffith say's "invites" you to introduce the vector potential.

HUGE { WELL DONE by me BIG!

Now idea is what is "physical" rules

so in Electrostatics

$$\vec{\nabla}\cdot\vec{E}=\rho/\varepsilon_0 \quad \& \quad \vec{\nabla}\times\vec{E}=0$$

most remain true!

so $\vec{E}=-\vec{\nabla}V$

cause 1) $\vec{\nabla}\times(-\vec{\nabla}V)=0$ (curl of grad = 0)

2) $\vec{\nabla}\cdot(\vec{\nabla}V)=\rho/\varepsilon_0$

$\nabla^2 V=\rho/\varepsilon_0$

just solve for V to make true & wha-la, you've got the $V(\vec{r})$ that makes

so the physical conditions 1&2 = Max's for static E fields "defines" V

life easier

& still keeps the "physical"

$$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{E}=\rho/\varepsilon_0\\ \&\ \vec{\nabla}\times\vec{E}=0\end{array}\right\}\text{True!}$$

Note V = Arbitrary

cause if $V'=V+C$

Then $\vec{\nabla}\times(-\vec{\nabla}V')=0$

and $\vec{\nabla}\cdot(-\vec{\nabla}V')=\nabla^2V'=\nabla^2V=\rho/\varepsilon_0$

Theories that have this Flexability are called "guages" ⟹ so V' also keeps the "physical" true!

So now we are invited to "figure" what a vector potential $\vec{A}$ should be

That maintains The physical

1.) $\vec{\nabla}\cdot\vec{B}=0$

2.) $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$

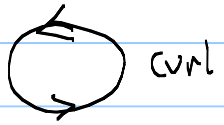
so we said let's define $\vec{A}(\vec{r})$ such that

$$\vec{B}(\vec{r})=\vec{\nabla}\times A(\vec{r})$$

OK check the physical to "define $\vec{A}$"

1.) $\vec{\nabla}\cdot\vec{B}=0$

well $\vec{\nabla}\cdot(\vec{\nabla}\times A(\vec{r}))=0$ Sure: div of a curl $=0$



now need

2.) $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$ ⇐ which will obviously be the one that defines $\vec{A}(\vec{r})$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\mu_0\vec{J}$$

$$(\vec{A}\times\vec{B}\times\vec{C})=BAC-CAB$$

RULE!

$$\vec{A}\times\vec{B}\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \underbrace{\vec{A}(\vec{\nabla}\cdot\vec{\nabla})}_{?} = \mu_0\vec{J}$$

well They, $\vec{\nabla}$, are just vectors, so Thin

$$\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\quad\right)$$

$$=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)=\text{scalar!}$$

Now, so it is a scalar

$$[\vec{A}, a] = 0$$

$$= \vec{A}a - a\vec{A} = 0$$

or $a\vec{A} = \vec{A}a$

so

$$\vec{\nabla}\times\vec{B} \qquad = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2_{x,y,z}\vec{A} = \mu_0\vec{J}$$

Note!

Laplacian

$$\nabla^2 V = \vec{\nabla}\cdot\vec{\nabla}V$$

$$\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}\right)\cdot\left(\frac{\partial V}{\partial x}\hat{i}+\frac{\partial V}{\partial y}\hat{j}\right)$$

$$=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}$$

$=$ scalar

But here we have

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{A} = \text{Vector!}$$

or

$$\nabla^2_{x,y,z}\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

not a dot product!

So $\nabla^2_{x,y,z}$ $\hat{o}$ on each component

$$\nabla^2_{xyz}A_x\hat{i} + \nabla^2_{xyz}A_y\hat{j} + \nabla^2_{xyz}A_z\hat{k}$$

So Be careful......

Finally we get

$$\boxed{\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2_{xyz}\vec{A} = \mu_0\vec{J}}$$

So again what we are saying is $\underline{\underline{\text{we want}}}$ an

$\vec{A}$ such that $\vec{B} = \vec{\nabla}\times\vec{A}$

& we are fitting this $\vec{A}$ by the physics

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \underline{0}$$ great

Now

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2_{xyz}\vec{A} = \mu_0\vec{J}$$

great! Just solve for $\vec{A}$ from this equation and it will all work!

So to get $\vec{A}$ solve $\underline{\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2_{xyz}\vec{A} = \mu_0\vec{J}}$

Not so easy to solve

But people are smart... here is the idea:

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

right?

well free to say $\vec{A}' = \vec{A} + \underbrace{\nabla Z}_{\text{gradient of another scalar}}$

cause

$$\nabla\times(A+\vec{\nabla}Z)$$

$$= \vec{\nabla}\times\vec{A} + \vec{\nabla}\times(\vec{\nabla}Z) \xrightarrow{} 0$$

$$= \vec{\nabla}\times\vec{A}$$

Just like $\underbrace{V' \to V + C}$ had no effect on

$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

$\vec{\nabla}\times\vec{E} = 0$

BUT with this freedom to add a $\vec{\nabla}Z$ to $\vec{A}$

you can always get a $\vec{\nabla}Z$ such that

$$\boxed{\vec{\nabla}\cdot\vec{A}} = \vec{\nabla}\cdot\underset{\vec{A}'}{(\vec{A}+\vec{\nabla}Z)} = \underbrace{\vec{\nabla}\cdot\vec{A} + \nabla^2 Z}\ \boxed{= 0}$$

$$\nabla^2 Z = \vec{\nabla}\cdot\vec{A}$$

just solve this Z

guaranteed such a solution

So can say you are looking for this

A' that has $\vec{\nabla}\cdot\vec{A} = 0$

Then: $\vec{\nabla}\times\vec{B} = \vec{\nabla}(\underbrace{\cancel{\vec{\nabla}\cdot\vec{A}'}}_{=0}) - \nabla^2\vec{A}' = \mu_0\vec{J}$

and our search reduces to

$$\nabla^2\vec{A}' = \mu_0\vec{J}$$

But since the $\vec{A}' = A' + \nabla Z$

The $\vec{\nabla}Z$ drops out in both

$\vec{\nabla}\cdot\vec{B} = 0$

& $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

Just say

$$\nabla^2\vec{A} = \mu_0\vec{J}$$

WOW! what we are doing is solving for Vector potential $\vec{A}$ in "The Coulomb" guage -- ie The arbitrariness of $\vec{A}$ that lets us Choose $\vec{\nabla}\cdot\vec{A} = 0$

Coulomb guage is starting pt for quantizing E&B field C14 in Griffs, Townsend & others!

Easiest, most common guage choise but there are others!

So: $\nabla^2 \vec{A} = \mu_0 \vec{J}$ ⇒'s 3 poisson equations which we know how to solve

$$\boxed{\begin{array}{l}\nabla^2 V = \frac{\rho_{free}}{\varepsilon_0} \\ \& \\ \nabla^2 V = 0 \quad \text{Laplaces}\end{array}}$$

$$\left(\nabla^2 A_x = \mu_0 J_x\right)\hat{x}$$

$$\left(\nabla^2 A_y = \mu_0 J_y\right)\hat{y}$$

$$\left(\nabla^2 A_z = \mu_0 J_z\right)\hat{z}$$

In any 1 direction

recall $\nabla^2 V = \frac{\rho_{free}}{\varepsilon_0}$

so

$$\Rightarrow V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,d\tau'}{\mathscr{r}}$$

ie $dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\mathscr{r}}$

$$A_x(r) = \frac{\mu_0}{4\pi}\int \frac{J_x(r')}{|\mathscr{r}|}\,d\tau'$$

or combining

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{|\mathscr{r}|}\,d\tau'}$$

So we then have:

$$\underbrace{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\,dl'}{|\mathscr{r}|} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'}{|\mathscr{r}|}}_{\text{for lines of current}}$$

$$\& = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{|\mathscr{r}|}\,da' \quad \text{for surface current!}$$

* Typically $\vec{A}$ follows the direction of $\vec{I}$

* again not as useful as $V(\vec{r})$

But plays HUGE role later
in chapter 10 Griffith
for energy in E&B field

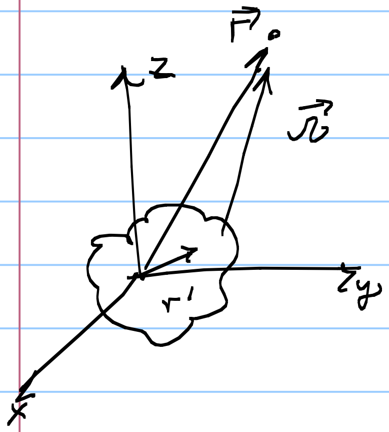
& Chpt 14 of Townsend!

Do the details of Gr. SS: Ex. 5.11 only

Now multipole expansion of $\vec{A}$

W/ Electric potential $V(r) = \frac{1}{4\pi\epsilon} \int \frac{\rho(r')\,d\tau'}{|\mathscr{r}|}$

we found it useful to look @ limit $\vec{r} \gg \vec{r}\,'$



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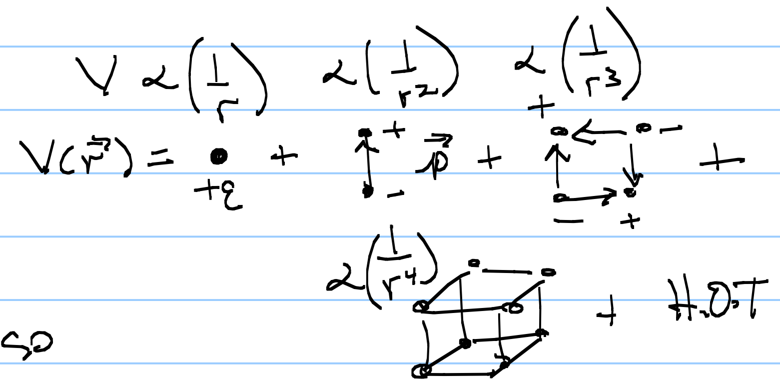


so

$$V(r) = \underset{\propto \frac{1}{r}}{\text{monopole}} + \underset{\propto \frac{1}{r^2}}{\text{dipole}} + \overset{\propto \frac{1}{r^3}}{\text{quadrupole}} + \text{H.O.T}$$

Gave a real physical feel for $V(r)$

& The Ability to approximate

$V(r)$ for REAL problems which are never exactly solvable

The heart of the expansion was in the $\frac{1}{|\mathscr{r}|}$ term in particular we found

$$\boxed{\frac{1}{|\mathcal{R}|}} = \frac{1}{\sqrt{r^2+(r')^2+2rr'\cos\theta}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

Love this result

Student re-do

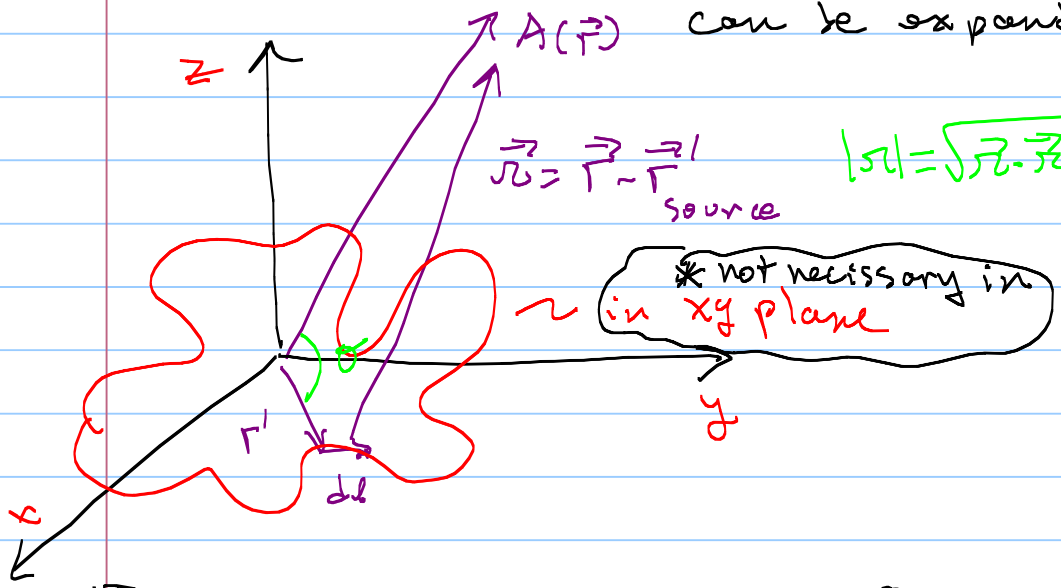
Legendre Polynomials = Complete Hilbert Space

Clue is.... azimuthal ϕ symmetry

Spherical problems

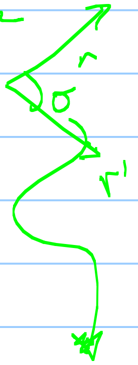
and clearly $\vec{r}$ is spherically symmetric in ϕ

So, for example: $\vec{A}(\vec{r})$ for a current loop can be expanded



$$|\mathscr{r}| = \sqrt{\mathscr{r}\cdot\mathscr{r}} = \left(r^2 + r'^2 - 2rr'\cos\theta'\right)^{1/2}$$

where $\theta' =$



$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{|\mathscr{r}|}\,dl' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n\cos(\theta')\,d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l}' + \frac{1}{r^2}\oint r'\cos\theta'\,dl' + \frac{1}{r^3}\oint r'^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)d\vec{l}' + H.O.T\right]$$

$$= \frac{\mu_0 I}{4\pi}\left[\propto\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + H.O.T\right]$$

monopole

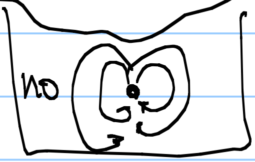
dipole

quadrupole } again

but clearly monopole term

$$\left.\frac{1}{r}\underbrace{\oint d\vec{l}}_{=0}\right\} = 0$$

ie – monopole term of $\vec{A}$ is always zero as there are no pt-like source of $\vec{B}$ (thus $\vec{A}$) & $\vec{\nabla}\cdot\vec{B}=0$



So ordinarily the dipole term dominates

so

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta' d\vec{l}' = \frac{\mu_0 I}{4\pi r^2}\underbrace{\oint(\hat{r}\cdot\vec{r}')d\vec{l}'}$$

which we can recognize!

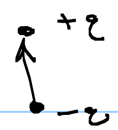
use eq 1.108 pg 57

$$\oint(\vec{c}\cdot\vec{r})d\vec{l}' = \vec{area}\times\vec{c}$$

so

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\vec{A}\times\hat{r}$$

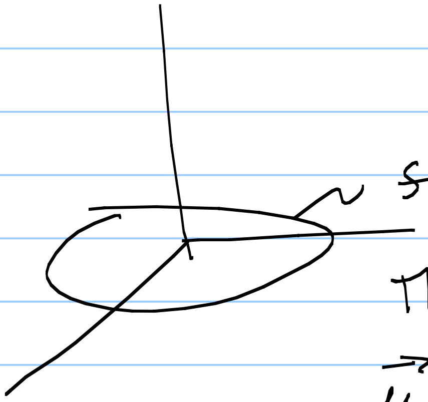
Now just @ $\vec{p}$ = Electric dipole =



we have $\vec{\mu}$ = magnetic dipole.

$$\vec{\mu} \equiv I \int d\vec{a}$$

is

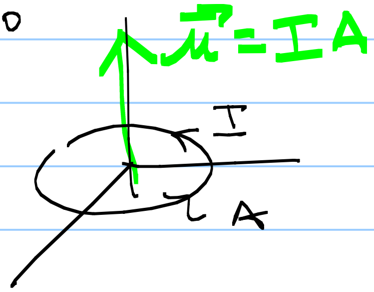


Then

$\vec{\mu} = IA_{loop}$ dir = Normal to area:

Now $\vec{\mu}$'s play same role as $\vec{p}$'s did for $\vec{E}$ in materials so KEY to understand

So



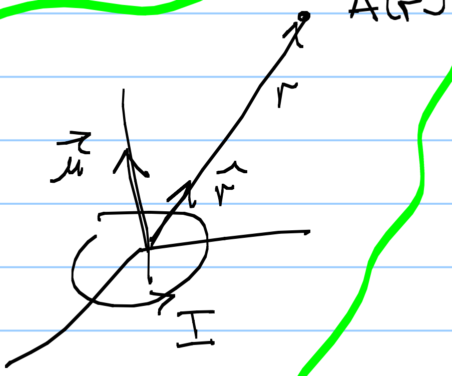
=

$\uparrow$ $\vec{\mu} = IA$

dir by RHR

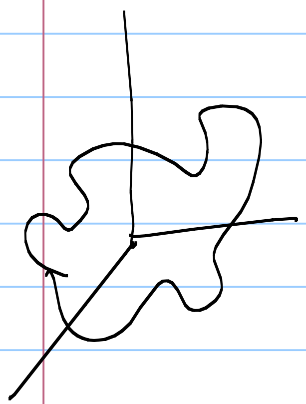
So

$$\vec{A}(\vec{r})_{dipole} = \frac{\mu_0}{4\pi} \frac{\vec{\mu} \times \hat{r}}{r^2}$$

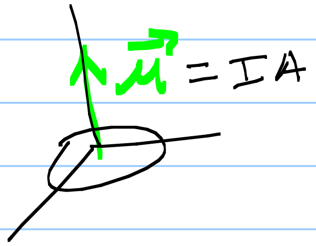


Note: That $\vec{A} \propto \vec{\mu} \times \hat{r}$ = along I

So



$\cong$



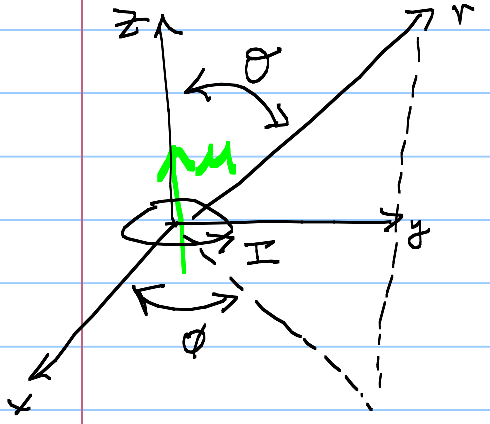
$+$ H.O.T.

$\Uparrow$

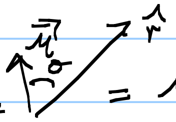
How hard do yo want to work?

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{\mu} \times \hat{r}}{r^2}$$

Now: what is $\vec{B}_{dipole}$



$\vec{\mu} \times \hat{r} =$



$= \mu \sin\theta \; \hat{\phi}$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{|\vec{\mu}||\hat{r}| \sin\theta}{r^2} \left(\hat{\phi}\right)$$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\mu \sin\theta}{r^2} \left(\hat{\phi}\right)$$

Now $\vec{B}_{dipole}(\vec{r}) = \vec{\nabla}_{spher} \times \vec{A}_{dipole}$

$\Leftarrow$ Students do

$$\vec{A}_{dip}(\vec{r}) = \left(\frac{\mu_0 \mu}{4\pi}\right)\left(\frac{\sin\theta}{r^2}\right)\hat{\phi}$$

So $\vec{B}_{dip}(\vec{r}) = \vec{\nabla}_{sph} \times \vec{A}(\vec{r}) =$

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, V_\phi) - \underbrace{\frac{\partial V_\theta}{\partial\phi}}_{\to 0}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\underbrace{\frac{\partial V_r}{\partial\phi}}_{\to 0} - \frac{\partial}{\partial r}(rV_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\underbrace{V_\theta}_{\to 0}) - \underbrace{\frac{\partial V_r}{\partial\theta}}_{\to 0}\right]\hat{\phi}$$

$$= \frac{\mu_0 \mu}{4\pi}\left[\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\sin\theta}{r^2}\right)\right]\hat{r} + -\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\sin\theta}{r^2}\right)\hat{\theta}\right]$$

$$\frac{1}{r^3\sin\theta}\left[\frac{\partial}{\partial\theta}[(\sin\theta)^2]\right] \qquad -\frac{1}{r}\frac{\partial}{\partial r}(\sin\theta\, r^{-1})$$

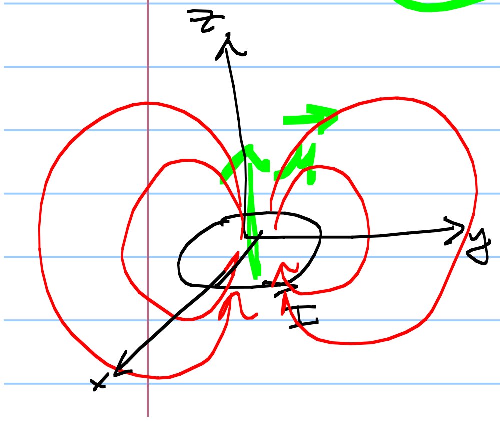
$$\frac{1}{r^3\cancel{\sin\theta}}\,2\cancel{\sin\theta}\cos\theta \qquad +\frac{\sin\theta}{r^3}\hat{\theta}$$

$$\frac{2\cos\theta}{r^3}\hat{r}$$

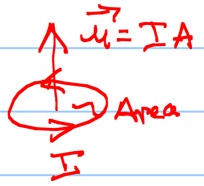
*

$$\boxed{\vec{B}_{dipole}(\vec{r}) = \frac{\mu_0 \mu}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]} \Leftarrow$$

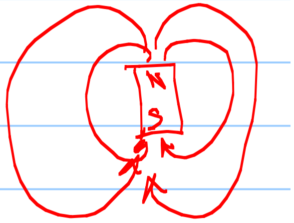
Azimuthally symmetric $\neq f(\phi)$



so $\uparrow \vec{\mu} =$

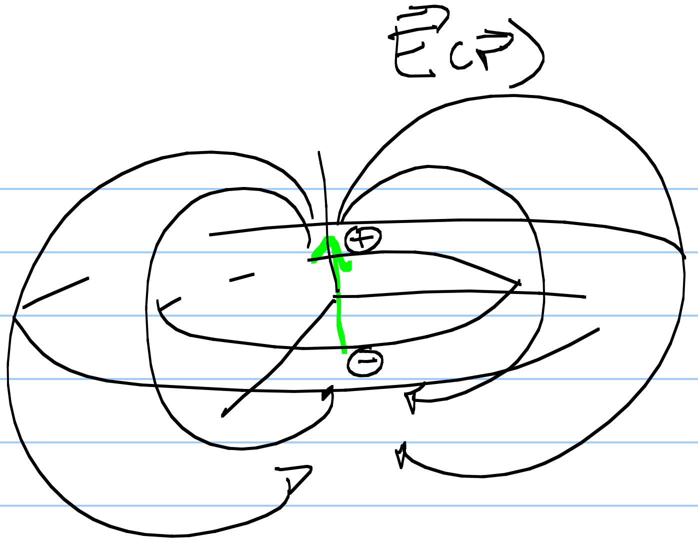


$\approx$

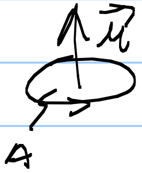


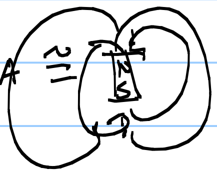
$\vec{\mu}$'s = little tiny magnets!

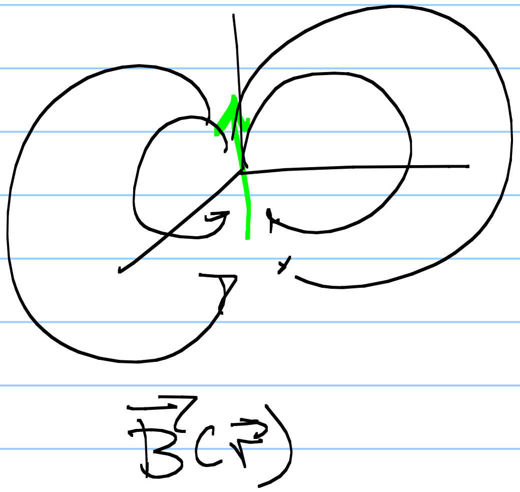
$\vec{p} = \overset{+}{\underset{-}{\uparrow}}\ q\vec{d}$



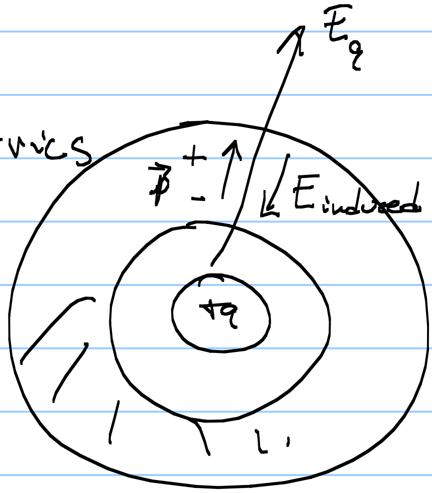
$\hat{\phi}$ symmetry

$\vec{\mu} =$


$= IA \simeq$




Again in dielectrics



So $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_r}\frac{\rho_{free}}{\varepsilon_0}$

ε_r ↔ dielectric constant

$\vec{M}$ will play similar role finding $\vec{B}$ in materials!