E.F. Deveney / BSC Physics PH438 Multipole Expansion of Electric Potential!

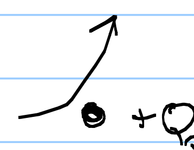
Note Title 11/20/2004

Multipole Expansion is my favorite example of 1.) a math technique
2.) general idea of how physicist solve REAL-problems!

Big idea:

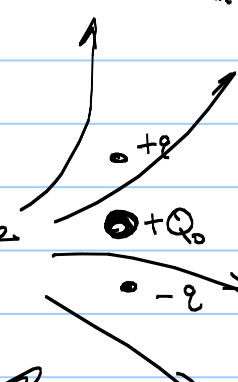
sub atomic particle don't much about

Low energy e^- looks like

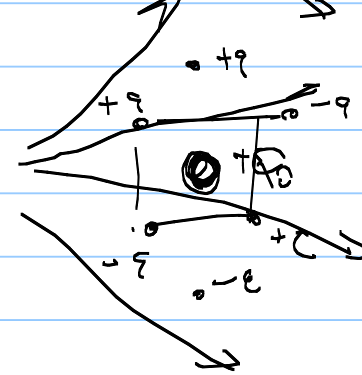


$\lambda = \frac{h}{p} = \frac{h}{mv}$

medium energy e^- looks like



High energy e^- looks like



idea is:

The harder you try (cost lot of effort to ↑ e^- beam energy)

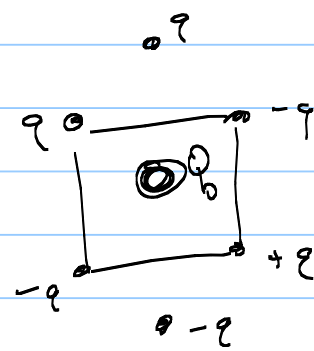
But The harder you look....
The more "precission" cross section $\perp$ $\underbrace{\Delta CS}_{\text{smaller \& smaller}}$

The more structure you see! or is revealed

(of course, perhaps,... is The structure is There)

WORKING The other way now

Say you know you have



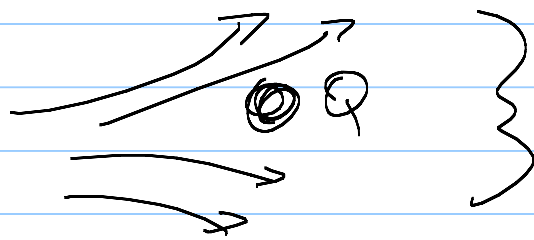
But you are doing

low energy Crystallography to study the conduction properties of This Thing in a Solid, say The Resistance of a wire.

MACROSCOPIC

Perhaps, This macro-property can

easily be approximated by

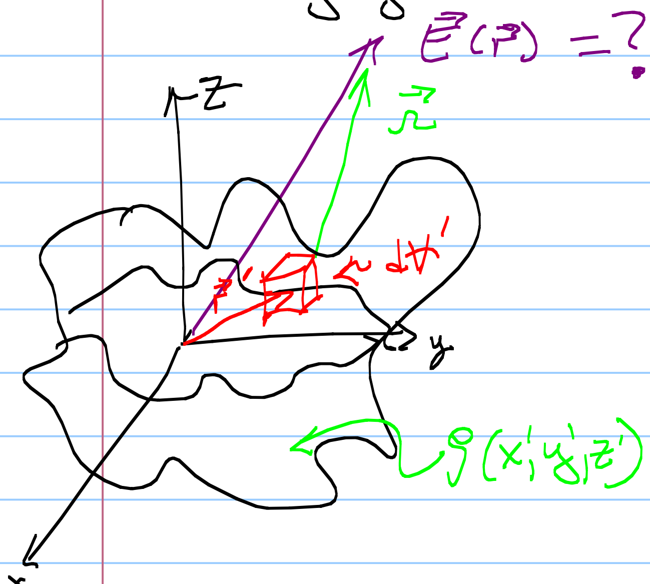


wow, Good macro Calc's are easier! and pretty good!

As a physicist you must have insights into Both

So multipole expansion of Electric potential is this scenario

Lets say you have BLOB of charge



Clearly

1.) $\vec{E}$ by brute force $\int\limits_{\tau} d\vec{E}(r)$
a mess

2.) No Symmetry to exploit

PLUS

3.) Since B.C.'s ARE a Mess
Little hope for Solving $\nabla^2 V = 0$ Laplaces
or
4.) Using Images?

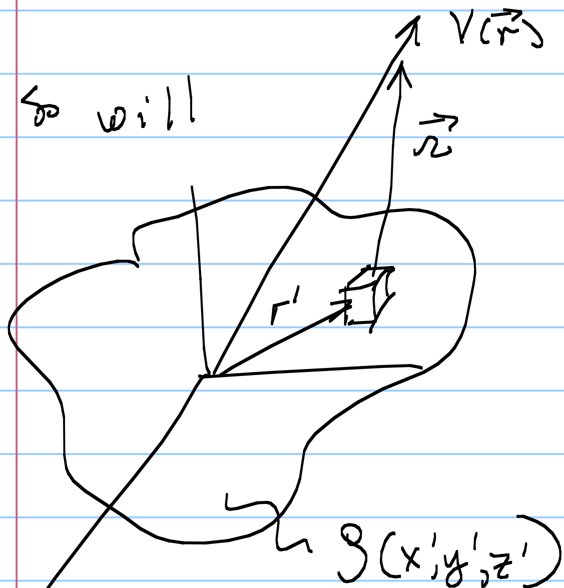
What to do?

Well Back to $V(\vec{r}) = \int\limits_{\tau} dV(r)$

at least $dV = \dfrac{k\,dq}{|\vec{\mathscr{r}}|}$; $\vec{\mathscr{r}} \equiv \vec{r}_{\text{field}} - \vec{r}\,'_{\text{source}}$

is a scalar.

so will

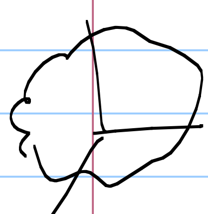
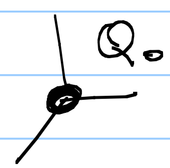


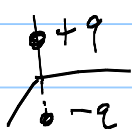
$$V(\vec r) = \int \frac{k\,dq}{|\vec{\mathscr{r}}|}$$

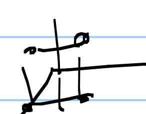
Then $\vec{E} = -\vec{\nabla} V(\vec r)$

and Find That *

* ie: depending on
1) How much work u want to do
2) How much you NEED to do based on desired, needed, precission

 $\cong$  is don't look so hard

(+)  is look bit harder

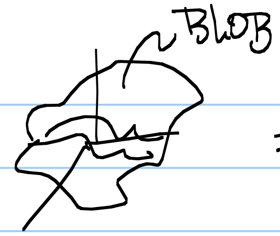
(+)  "

(+)  "

you can build up your answer!

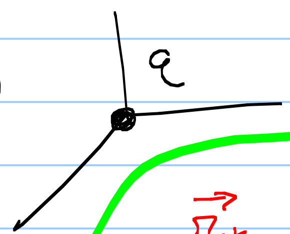
// IT is a physical ∞ series / expansion.

So

$V(\vec{r})$ for  $= V(Q_0) + V\begin{pmatrix} +q \\ -q \end{pmatrix} + V\begin{pmatrix} + & - \\ - & + \end{pmatrix} + \cdots$

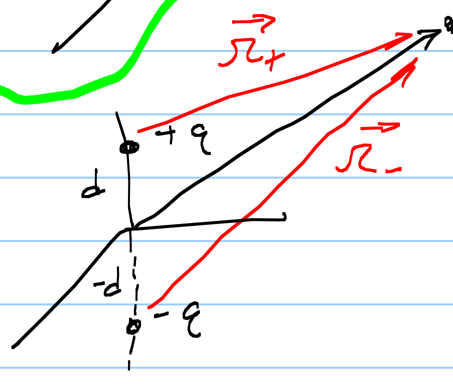
pot due to Electric monopole — " " dipole — " " quadrupole $+ \cdots \infty$

So RECALL

1.) 

$V(r) = \dfrac{Kq}{r}$: $V \propto \dfrac{1}{r}$

Electric monopole

2.) 

$V(\vec{r})$ = Electric dipole

$$V(\vec{r}) = K\left(\frac{q}{|\Omega_+|} + \frac{-q}{|\Omega_-|}\right)$$

$$\vec{\Omega}_+ = \vec{r} - \vec{r}'_+ = \vec{r} - \vec{d}$$

$$\vec{\Omega}_- = \vec{r} - \vec{r}'_- = \vec{r} - (-\vec{d}) = \vec{r} + \vec{d}$$

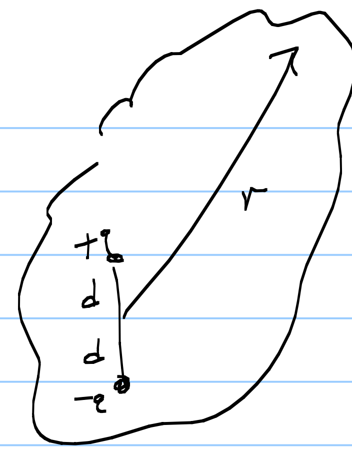
now $|\Omega_+| = \sqrt{\vec{\Omega}_+ \cdot \vec{\Omega}_+} = \left(r^2 + d^2 - rd\cos\theta\right)^{\frac{1}{2}}$

$|\Omega_-| = \sqrt{\vec{\Omega}_- \cdot \vec{\Omega}_-} = \left(r^2 + d^2 + rd\cos\theta\right)^{\frac{1}{2}}$

But clearly $\cos\theta_+ = \cos\theta_- = \cos(\theta)$

so $\dfrac{1}{|\mathscr{r}_\pm|} = \dfrac{1}{\left(r^2 + d^2 \mp rd\cos\theta\right)^{\frac{1}{2}}}$

look for $r \gg d$ ie



$$\left[r^2\left(1 + \frac{d^2}{r^2} \mp \frac{d}{r}\cos\theta\right)\right]^{-1/2}$$

for $r \gg d$ drops

$$\simeq \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-\frac{1}{2}}$$

WOW!
did you
Catch that!
BSE!

$$\simeq$$

$$\frac{1}{|\mathscr{r}_\pm|} = \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

so

$$V(\vec{r}) = kq\left[\frac{1}{|\mathscr{r}_+|} - \frac{1}{|\mathscr{r}_-|}\right]$$

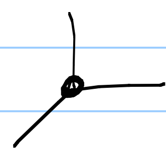
$$\simeq kq\left[\frac{1}{r}\left(1 + \frac{d}{2r}\cos\theta\right) - \frac{1}{r}\left(1 - \frac{d}{2r}\cos\theta\right)\right]$$

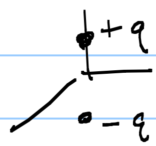
$$\simeq kq\,\frac{1}{r}\left(\frac{2d}{2r}\cos\theta\right)$$

$$V_{dipole}(\vec{r}) \cong K \frac{qd\cos\theta}{r^2} \quad \text{or} \quad V_{dipole} \propto \frac{1}{r^2}$$

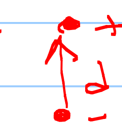
Now similarly could do for Electric Quadrupole & Octupole & get

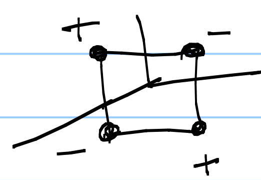
Electric:

monopole  $V_m \propto \frac{1}{r}$

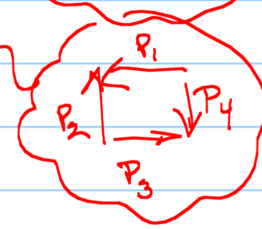
dipole 

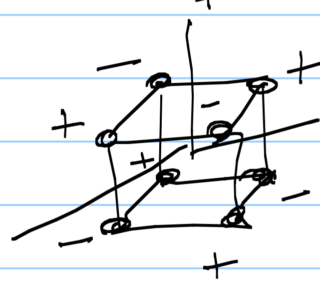
$V_d \propto \frac{1}{r^2}$

*E-dipole defined as $\vec{p} = q\vec{d}$ 

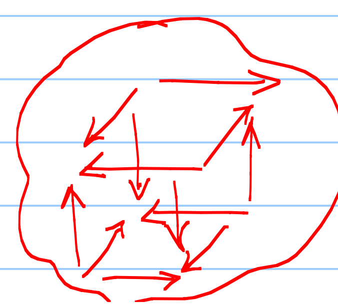
quadrupole 

$V_q \propto \frac{1}{r^3}$



Octopole 

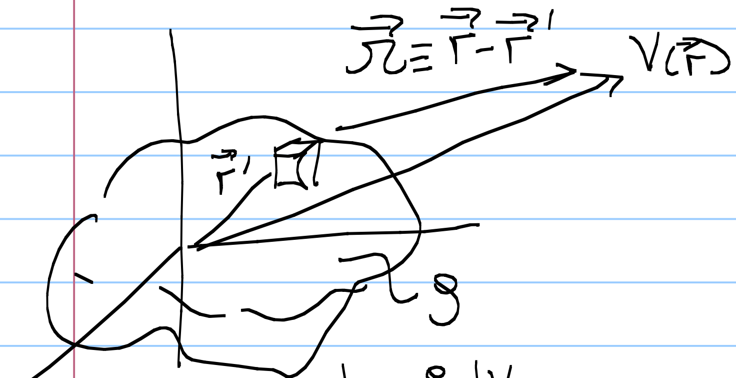
$V_o \propto \frac{1}{r^4}$



so on!

With our Electric "poles" in hand... finally to multipole expansion

Now FORMALLY RIGOROUSLY!



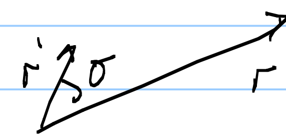
$dq = \rho\, d\tau$

$$dV(\vec{r}) = \frac{k\,dq}{|\mathscr{r}|} \quad ; \vec{\mathscr{r}} = \vec{r} - \vec{r}'$$

$$|\vec{\mathscr{r}}| = \sqrt{(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')}$$

$$= \left(r^2 + r'^2 - 2rr'\cos\theta\right)^{1/2}$$

where



now looking for $|r| \gg |r'|$

i.e.

$$|\vec{\mathscr{r}}| = r\left(1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta\right)^{\frac{1}{2}}$$

Let assume ↑ is pretty small so

$$= r(1+\varepsilon)^{1/2} \quad \text{where } \varepsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta\right)$$

so condition $|r| \gg |r'|$

$$\Rightarrow \text{is } \varepsilon \ll 1$$

great! $(1+\varepsilon)^{1/2}$ now ready for BSE

AS Griffiths says.... this INVITES one to use BSE

$$V(\vec{r}) = \int dV(\vec{r}) = \int \frac{k\, dq}{r(1+\varepsilon)^{1/2}} = \int_{\mathcal{V}} \frac{k\, \rho(r')\, d\tau'}{r(1+\varepsilon)^{1/2}}$$

where $\dfrac{1}{|\mathscr{r}|} = \dfrac{1}{r}(1+\varepsilon)^{-1/2} = 1 + m\varepsilon + \dfrac{m(m-1)}{2}\varepsilon^2 + \dfrac{m(m-1)(m-2)}{3!}\varepsilon^3 +$ H.O.T.

$$= 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots$$

sticking w/ $\quad$ & $\quad \varepsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta\right)$

$$\frac{1}{|\mathscr{r}|} = \frac{1}{r}\Bigg[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta\right)^2 + -\frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta\right)^3 + \cdots\Bigg]$$

Now H.W. $\Rightarrow$ Expand & Collect terms of $\left(\frac{r'}{r}\right)$ to get

$$\frac{1}{|\mathscr{r}|} = \frac{1}{r}\Bigg[$$

$$\frac{1}{|\mathscr{r}|} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\theta) + \left(\frac{r'}{r}\right)^2 (3\cos^2\theta - 1)\frac{1}{2} + \left(\frac{r'}{r}\right)^3 (5\cos^3\theta - 3\cos\theta)\frac{1}{2} + \cdots\right]$$

& WO-AH!

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \underbrace{P_n(\cos\theta)}$$

Legendre Polynomials again!

→ showed up as "seperable" solution for $\Theta(\theta)$ for $\nabla^2 V = 0$ in Spherical Coords!

Wild

$$\frac{1}{\text{Separation Vector}} = \frac{1}{r}\sum \text{series expan of L.P.'s}$$

CRAZY!

L.P.'s $P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = (3x^2 - 1)\frac{1}{2}$

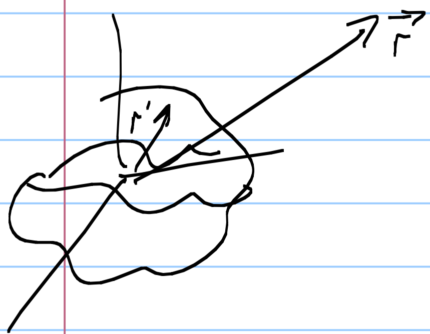
⋮

L.P.'s = Complete & $\perp$ Function space (ie Hilbert space) build from x^n's which are Complete set $\neq \perp$

recall Rodrigues Formula to 'generate' L.P.s

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2 - 1)^\ell$$

So ....



$|r| >> |r'|$

Then

$$V(\vec{r}) = K \int_{v} \frac{dq}{|\mathcal{R}|} = K \int \frac{\rho(r')\,dv'}{|\mathcal{R}|}$$

$$= K \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos\theta)\, \rho(r')\, dv'$$

OR

$n=0$, $P_0(\cos\theta) = 1$ $\quad$ $n=1$, $P_1(\cos\theta) = \cos\theta$ $\quad$ $n=2$, $P_2 = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$

$$V(\vec{r}) = K \left[\frac{1}{r} \int \rho(r')\, dv' + \frac{1}{r^2} \int r' \cos\theta\, \rho(r')\, dv' + \frac{1}{r^3} \int (r')^2 \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \rho(\vec{r}')\, dv' + H.O.T \right]$$

NOTE

$\frac{1}{r}$ mono-pole term

$\frac{1}{r^2}$ dipole term

$\frac{1}{r^3}$ quadrupole! term!

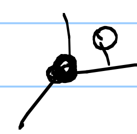
This is it! This is our multipole expansion of $V(r)$ in terms of power of $\frac{1}{r}$

IT IS EXACT But power in use as approx too!

Let's see...

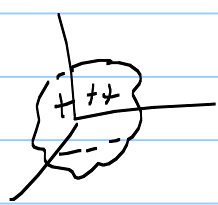
1) ORDINARILY @ large $|r \gg r'|$, expansion dominated by 0^{th} order term or monopole

$$V(\vec{r}) \cong K \frac{1}{r} \underbrace{\int_{\mathcal{V}} \rho(\vec{r}\,')\, d\tau'}_{Q} = K \frac{Q_{TOT}}{r} \Bigg\} \text{ cool}$$



* if, Thats all There is, H.O.T will vanish exactly!

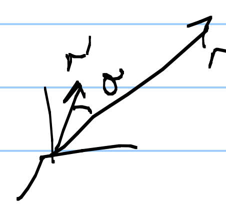
2) Now what is $Q_{TOT} = 0$, but it is distributed non-uniformly!



Then Dipole Term will vanish (unless it too vanishes)

$$V(\vec{r}) \cong \underbrace{K \frac{1}{r} \int_{\mathcal{V}} \rho(\vec{r}\,')\, d\tau'}_{\text{monopole} \to 0,\ Q_{TOT} = 0} + \underbrace{K \frac{1}{r^2} \int_{\mathcal{V}} r' \cos\theta\, \rho(\vec{r}\,')\, d\tau'}_{\text{(Dipole Term)}}$$

now recall

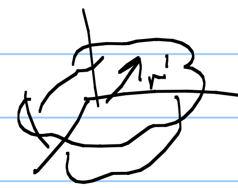


so $\vec{r} \cdot \vec{r}\,' = |r||r'| \cos\theta$ $\therefore$

$\hat{r} \cdot \vec{r}\,' = (1)\, r' \cos\theta$

$$V_{\text{dipole}}(\vec{r}) = K \frac{1}{r^2} \hat{r} \cdot \underbrace{\int_{V} \vec{r}\,' \rho(\vec{r}\,')\, dV'}$$

depends only on source ... i.e. integrate over r'



so Thus Term is SOLELY property of the Source BLOB itself & is use fully called

determined essentially by geometry size, shape & geometry

$$\vec{p} = \int \vec{r}\,' \rho(\vec{r}\,') dV' = \text{Dipole moment of the charge distribution!}$$

$\vec{p} = \sum_i q_i r_i' =$ general BUT

essentially $\vec{d} = q\vec{d} =$ 'physical' dipole $+q$ / $-q$

where $+q$ is @ r_+ & $-q$ @ r_-

$\vec{P} = q r_+' - q r_-'$

cool!

$+q$, r_+', r_-', $-q$

$\vec{p} = q(\uparrow - (\downarrow))$

$= q(\uparrow_{r_+} \uparrow_{-r_-}) = q\vec{d}_{\pm \text{ to}}$ so

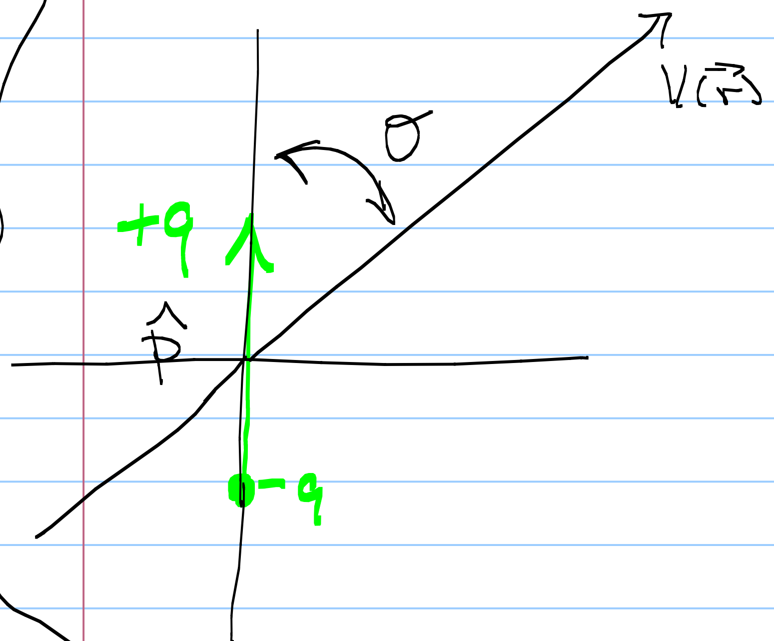
due to source!

$$V_{\text{dipole}}(\vec{r}) = K \frac{\vec{p} \cdot \hat{r}}{r^2}$$

due to field position!

$-r_-'$, r_+' $= +\vec{d}$ from $-q$ to $+q$

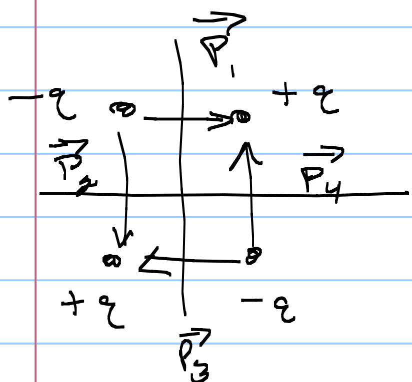
So $V_{dipole}(\vec{r}) = \dfrac{K\vec{p}\cdot\hat{r}}{r^2} = \dfrac{Kp\cos\theta}{r^2}$



HUGE!

Now $\vec{E}$ - dipoles, $\vec{p}$, are vectors, so add like vectors.

So for example



$$\Rightarrow \left.\begin{matrix} \rightarrow P_1 + \leftarrow P_3 \\ + \downarrow P_2 + P_4 \uparrow \end{matrix}\right\} = P_{TOT} = 0$$

$\therefore$

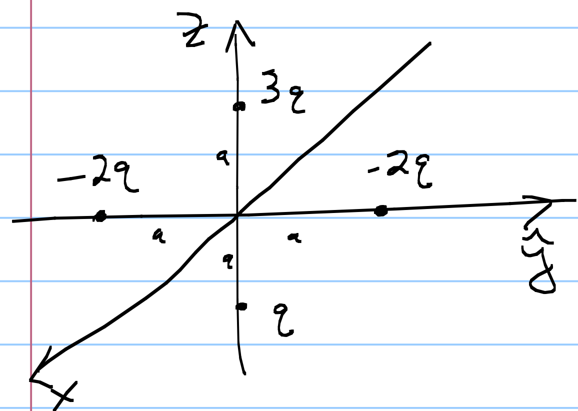
$$V(r) = \underbrace{\text{monopole}}_{\to 0} + \underbrace{\text{dipole}}_{\to 0} + \underbrace{\text{quadrapole}}_{\propto \frac{1}{r^3}}$$

here $V(r) \propto \frac{1}{r^3}$

$V(r)$ is dominated by $\frac{1}{r^3}$

"if looked closely could see individual dipoles"

H.W. Prob 3.27



in general

$\vec{p} = \sum_i q_i r_i'$

Then here

$$p\,\hat{z} = \left[3q(+a) + 1q(-a)\right] = 2qa\,\hat{z}$$

$$p\,\hat{y} = \left[-2q(a) + -2q(-a)\right] = 0\,\hat{y}$$

$\therefore$

V good think $\propto \frac{1}{r} + \frac{1}{r^2}$

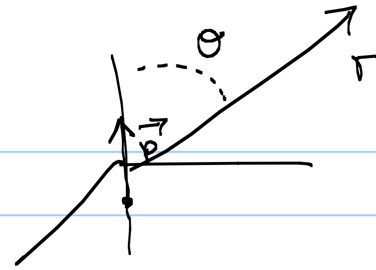
$$K\frac{4Q}{r} + \frac{K\,\vec{p}\cdot\hat{r}}{r^2}$$

$$V(r) = K\left[\frac{4Q}{r} + \frac{2qa\,\hat{z}\cdot\hat{r}}{r^2}\right]$$

Gauss doesn't include this!

$\boxed{\text{Now do H.W. Prob 3.30 a}}$

FINALLY!



$\vec{E}_{dipole} = ?$

$$\vec{E}_d(\vec{r}) = -\vec{\nabla} V_d(\vec{r})$$

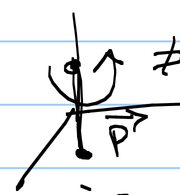
$$V_{dipole}(r,\theta) = k\frac{\hat{r}\cdot\vec{p}}{r^2} = k\frac{p\cos\theta}{r^2}$$

$$\vec{E}_d(\vec{r}) = -\vec{\nabla}_{spher} V_d(\vec{r}) = E_r\,\hat{r} + E_\theta\,\hat{\theta} + E_\phi\,\hat{\phi}$$

$$E_r = \frac{-\partial V}{\partial r} = k\frac{2p\cos\theta}{r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = k\frac{p\sin\theta}{r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0 \Leftarrow \text{no } f(\phi)$$



i.e azimuthal symmetry

$$\vec{E}_{dipole}(r,\theta) = k\frac{p}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

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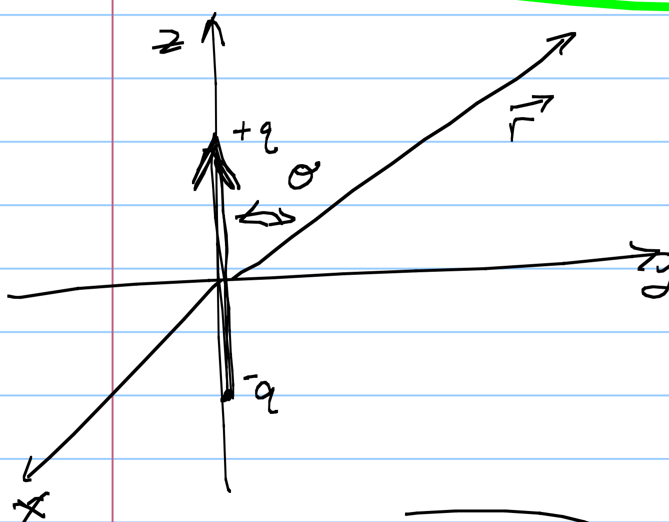
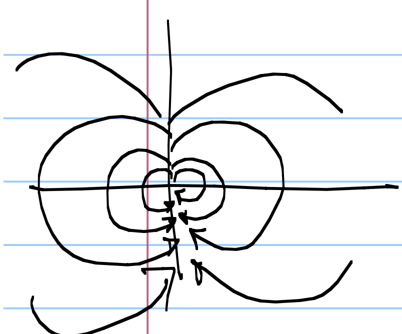
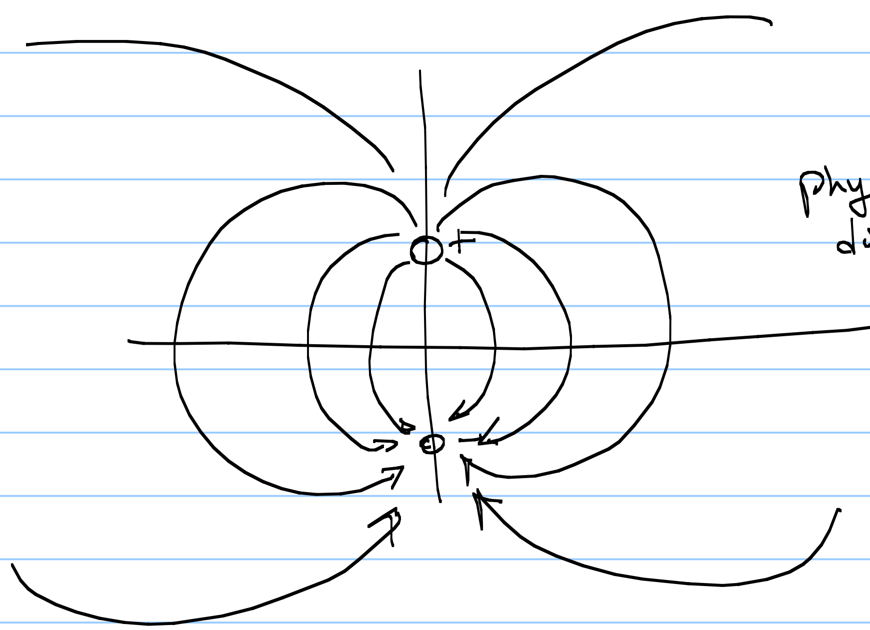


has azimuthal symmetry

so

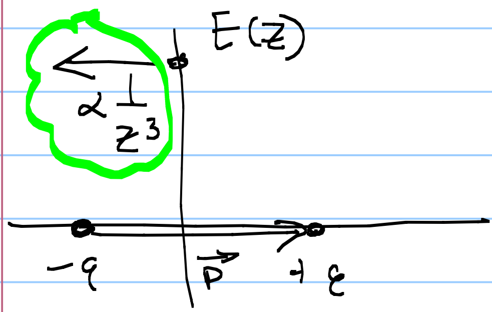


pure dipole!



physical dipole

Now we solved

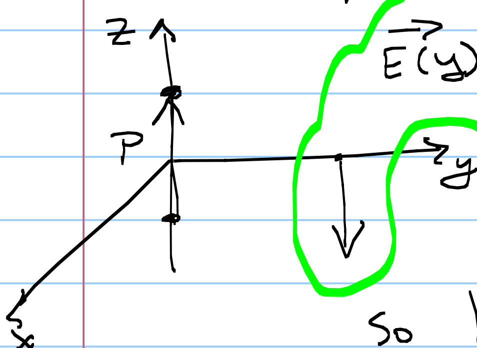


& got $\vec{E}(z) \propto \frac{1}{z^3}(-\hat{y})$

So is all cool!

So for ↓

expect $\vec{E}(y) \propto \frac{1}{z^3}$ in $\hat{z}$ dir

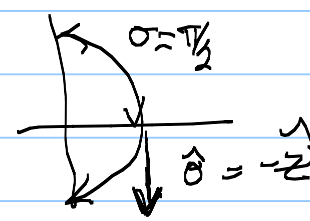


So here θ = polar = $\pi/2$

& ϕ = azimuthal = $\pi/2$

$$\therefore \vec{E}_{dipole}(y, \pi/2) = \frac{kP}{y^3}\left(2\underbrace{\cos(\pi/2)}_{\to 0}\hat{r} + \sin(\pi/2)\,\hat{\theta}\right)$$

$$= \frac{kP}{y^3}\,\hat{\theta} \quad (\text{at } x = z = 0) = -\hat{z}$$



yes!

So

$$E = k\frac{qd}{y^3}\,-\vec{z}$$