Multipole Expansion is my favorite example of:
1) a math technique
2) general idea of how physicists solve REAL problems!

Big idea:

subatomic particle don’t much about

\[ Z = \frac{\hbar}{\ell} = \frac{\hbar}{mv} \]

low energy \( e^- \) \( e^- e^- e^- \) looks like \( +Q \)

medium energy \( e^- \) \( e^- e^- e^- \) looks like \( +Q \)

high energy \( e^- \) \( e^- e^- e^- \) looks like

idea is:
The harder you try (cost lot of effort) to \( e^- \) become \( e^- \)
But the harder you look,...

The more "precession" cross section is ACS smaller & smaller

The more structure you see! or is revealed

(of course, perhaps,... is the structure is thus)

WORKING The other way now

Say you know you have

But you are doing

Low energy crystallography to study the conductive properties of this thing in a solid, say the resistance of an wire.

Perhaps, this macro-property can
easily be approximated by

wow, Good macro calcs are easier! and pretty "good!"

As a physicist you must have insights into both
So multiple expansion of Electric potential is this scenario

Let's say you have BLOB of charge \( \mathbf{E}(P) = ? \)

Clearly

1) \( \mathbf{E} \) by brute force \( \int \mathbf{E} \cdot d\mathbf{S} \) a mess

2) No symmetry to exploit \( \text{PLUS} \)

3) Since B.C.'s ARE a mess
Little hope for solving \( \nabla^2 V = 0 \) laplace's

4) Using Images?

What to do?

Well Back to \( V(P) = \int \mathbf{E} \cdot d\mathbf{l} \)
at least \( dV = \frac{kde}{|r|} \)

is a scalar.

\[ \nu(\vec{r}) \]

will

\[ \nu(\vec{r}) = S \frac{kde}{|r|} \]

Then \( \vec{c} = -\nabla \nu(\vec{r}) \)

\[ \mathbf{S}(x',y',z') \]

and Find That.

\[ \mathbf{S} = \mathbf{Q} - \mathbf{Q} \]

\[ \frac{\mathbf{S} + \mathbf{Q}}{\mathbf{Q} - \mathbf{Q}} \]

is don't look so hard

is look bit harder your

\[ \text{It is a physical } \infty \text{ series/ expansion.} \]

\[ \text{ie: depending on} \]

1) How much work u want to do

2) How much you NEED to do based on desired needed precision

\[ \text{you can build up} \]

\[ \text{your answer!} \]
So

\[ V(\mathbf{r}) = V(\mathbf{r}_0) + V(\mathbf{r}_{-\mathbf{e}}) + V(\mathbf{r}_{+\mathbf{e}}) + \cdots \]

\[ \text{potential due to} \]

- Electric monopole
- Dipole
- Quadrupole

\[ \mathbf{V}(\mathbf{r}) = \frac{k \mathbf{e}}{r} \]

Electric dipole

\[ V(\mathbf{r}) = k \left( \frac{\mathbf{e}}{|r|^3} + \frac{-\mathbf{e}}{|r|^3} \right) \]

\[ \mathbf{r}_+ = \mathbf{r}_+ - \mathbf{r}_- = \mathbf{r}_+ - \mathbf{r}_- \]

\[ \mathbf{r}_- = \mathbf{r}_+ - \mathbf{r}_- = \mathbf{r}_+ - (\mathbf{r}_+ + \mathbf{r}_-) = \mathbf{r}_- + \mathbf{r}_+ \]

Now

\[ |\mathbf{r}_+| = \sqrt{\mathbf{r}_+ \cdot \mathbf{r}_+} = \sqrt{r^2 + d^2 - rd \cos \theta} \]

\[ |\mathbf{r}_-| = \sqrt{\mathbf{r}_- \cdot \mathbf{r}_-} = \sqrt{r^2 + d^2 + rd \cos \theta} \]

But clearly \( \cos \theta_+ = \cos \theta_- = \cos \theta \)
\[
\frac{1}{|\mathbf{r}|} = \frac{1}{\left( r^2 + \frac{d^2}{r^2} + rd \cos \theta \right)^{\frac{1}{2}}}
\]

Look for \( r \gg d \), i.e.

\[
\left[ r^2 \left( 1 + \frac{d^2}{r^2} - \frac{d}{r} \cos \theta \right) \right]^{\frac{1}{2}}
\]

So \( r \gg d \) drops

\[
\frac{1}{|\mathbf{r}|} = \frac{1}{\frac{r}{\left( 1 + \frac{d}{2r} \cos \theta \right)}^{\frac{1}{2}}}
\]

\[
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\]

So

\[
\sqrt{\langle \mathbf{r}^2 \rangle} = K \frac{q}{\left[ \frac{1}{|\mathbf{r}|} - \frac{1}{|\mathbf{r}_L|} \right]^{\frac{1}{2}}}
\]

\[
\approx K q \left[ \frac{1}{r} \left( 1 + \frac{d}{2r} \cos \theta \right) - \frac{1}{r} \left( 1 - \frac{d}{2r} \cos \theta \right) \right]
\]

\[
\approx K q \frac{1}{r} \left( \frac{2d}{2r} \cos \theta \right)
\]
$$V(\vec{r}) = k \frac{q_0 \cos \theta}{r^2} \quad \text{or} \quad V_{dipole} \propto \frac{1}{r^2}$$

Now similarly could do for \textit{Quadrupole} & \textit{Octopole} & get

\begin{align*}
\text{Electric:} \\
\text{monopole} & \quad V_m \propto \frac{1}{r} \\
\text{dipole} & \quad V_d \propto \frac{1}{r^2} \\
\text{quadrupole} & \quad V_q \propto \frac{1}{r^3} \\
\text{Octopole} & \quad V_o \propto \frac{1}{r^4} \\
\end{align*}

so on!
With our Electric "poles" in hand... Now formally rigorously!

\[ \mathbf{\mathbf{E}} = \mathbf{E}' \] \( V(r) = \frac{k e}{|r - r'|} \)

\[ r = \sqrt{(r - r')^2} \]

\[ = \left( r^2 + r'^2 - 2rr' \cos \theta \right)^{1/2} \]

where \( r' \rightarrow r \)

now looking

Set \( |r| >> |r'| \)

\[ |r - r| = r \left( 1 + \left( \frac{r'}{r} \right)^2 - 2 \left( \frac{r'}{r} \right) \cos \theta \right)^{1/2} \]

let assume \( \theta \) is pretty small so

\[ = r \left( 1 + \varepsilon \right)^{1/2} \text{ where } \varepsilon = \left( \frac{r'}{r} \right) \left( \frac{r'}{r} - 2 \cos \theta \right) \]

So conditioning \( |r| >> |r'| \)

\[ \Rightarrow \varepsilon \ll 1 \]

Great! \( (1 + \varepsilon)^{1/2} \) now ready for BSE
\[ V(r^2) = S_v V(r^2) = \sum \kappa \frac{d}{r(1+\varepsilon)^2} = \sum \kappa \frac{8r(1+\varepsilon)^3}{r^2} \]

Where
\[
\frac{1}{|v|} = \frac{1}{r} \left(1 + \varepsilon\right)^{-1} = 1 + mE + \frac{m(m+1)\varepsilon^2}{2} - \frac{m(m+1)(m-2)\varepsilon^3}{3} + \ldots
\]

\[
\frac{1}{|v|} = 1 - \frac{1}{2} \varepsilon + \frac{3}{8} \varepsilon^2 - \frac{5}{16} \varepsilon^3 + \ldots
\]

\[ E = \left(\frac{r}{r_1}\right)^2 \left(\frac{r_1}{r} - 2\cos\theta\right) \]

\[ \frac{1}{|v|} = \frac{1}{r} \left[1 - \frac{1}{2} \left(\frac{r_1}{r}\right)^2 \left(\frac{r_1}{r} - 2\cos\theta\right)^2 + \frac{3}{8} \left(\frac{r_1}{r}\right)^2 \left(\frac{r_1}{r} - 2\cos\theta\right)^2 + \ldots\right] \]

Now H.W. \implies Expand \& Collect terms to get

\[ \frac{1}{|v|} = \frac{1}{r} \left[ \ldots \right] \]
\[ \frac{1}{|\mathbf{r}|} = \frac{1}{r} \sqrt{1 + \left( \frac{\mathbf{r}_\perp}{r} \right)^2 \left( 3 \cos^2 \sigma - 1 \right) + \cdots} \]

\[ \pm 1 = \frac{1}{r} \sum_{n=0}^{\infty} \left( \frac{r_r}{r} \right)^n P_n(\cos \sigma) \]

Legends Polynomials

Wild

Separation Vector

Series of LP's

CRAZY!

Recall Rodrigues formula to generate LP's:

\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \]

\text{LP's showed up as}

"Separable" solution to \( \nabla^2 \psi = 0 \)

in Spherical COords!

LP's:

\[ P_0(x) = 1 \]

\[ P_1(x) = x \]

\[ P_2(x) = (3x^2 - 1)^{1/2} \]

LP's = Complete and Function space (i.e. Hilbert space)

build from \( x^n \)'s which are complete set
\[ V(r) = \kappa \sum_{n=0}^{\infty} \frac{r^n P_n(\cos \phi) \sin \phi}{n!} \]

OR

\[ V(r) = \kappa \left[ \frac{1}{r} \left( \frac{8r}{3} \right)^{1/2} + \frac{1}{r^2} \left( \frac{8r}{3} \right)^{1/2} \right] + \frac{r^2}{f^3} \left( \frac{10}{3} \right)^{1/2} \]

**NOTE**

- \( \frac{1}{r} \) term: monopole term
- \( \frac{1}{r^2} \) term: dipole term
- \( \frac{r^2}{f^3} \) term: quadrupole term!

This is it! This is our multipole expansion of \( V(r) \) in terms of power of \( \frac{1}{r} \).

\[ IT \text{ IS EXACT BUT power in use as approx too!} \]
1) Let's see... ordinarily for large \( r \gg r' \), expansion dominated by 0th order term on monopole

\[
V(r') = K \int \frac{g(r')}{r'} \; d\mathbf{r}' = K \frac{Q_{\text{tot}}}{r'}
\]

... that's all. There is, at O(1), will vanish exactly!

2) Now what if \( Q_{\text{tot}} = 0 \), but it is distributed non-uniformly!

Then dipole term will vanish (unless it too vanishes)

\[
V(r) \approx K \frac{1}{r} \int g(r') \; d\mathbf{r}' + K \frac{1}{r^2} \int \cos \theta g(r') \; d\mathbf{r}'
\]

\[
Q_{\text{tot}} = 0
\]

now recall

\[
\mathbf{r} \cdot \hat{r'} = |\mathbf{r}||\mathbf{r}'| \cos \theta \approx 0
\]

\[
\mathbf{r} \cdot \hat{r'} = (1) r' \cos \theta
\]
\[ V_{\text{dipole}}(r) = \kappa \cdot \left( \sum_{r'} \mathbf{r} \cdot \mathbf{a}_{r'} \right) \]

This term is a non-uniform property of the source. Integrate over \( r' \).

So depends only on source... i.e.

\[ \mathbf{P} = \sum_{r'} \mathbf{a}_{r'} \]

\( \mathbf{P} = \mathbf{e}_0 \mathbf{r} \) is a general but

usefully called dipole moment.

\( \mathbf{P} \) is essentially due to source!

\[ \nabla \cdot \mathbf{P} = \kappa \cdot \mathbf{r} \]

\( \nabla \cdot \mathbf{P} \) is due to field position!

\[ \mathbf{P} \] is determined essentially by geometry.

Shape size.
So \( V_{\text{dipole}}(\vec{r}) = \frac{k \vec{p} \cdot \hat{r}}{r^2} = \frac{k p \cos \theta}{r^2} \).
New dipole $\vec{p}$, one vector, so add little vectors.

So for an example

\[
\begin{align*}
\vec{P}_1 + \vec{P}_2 - \vec{P}_3 + \vec{P}_4 & \Rightarrow \quad \vec{P}_1 + \vec{P}_3 + \vec{P}_2 + \vec{P}_4 = \vec{P}_{\text{tot}} = 0 \\
\end{align*}
\]

\[
\text{V}(r) = \text{monopole} + \text{dipole} + \text{quadrupole}
\]

\[
\text{here } V(r) \propto \frac{1}{r^3}
\]

\[
\text{V}(r) \text{ is dominated by } \frac{1}{r^3}
\]

"If looked closely could see individual dipoles"
H.W. Prob 3.27

\[ \mathbf{B} = \sum_{i} q_i \mathbf{r}_i' \]

Then here

\[ \mathbf{P} \mathbf{z} = [3q(z+2) + q(z-2)] = 2qz \mathbf{z} \]

\[ \mathbf{P} \mathbf{y} = [2q(z)+ -2q(z-2)] = 0 \mathbf{y} \]

\[ V \text{ good想法} < \frac{1}{r} + \frac{1}{r^2} \]

\[ k \frac{4\pi}{r} + k \frac{3\pi^2}{r^2} \]

\[ V(r) = k \left[ \frac{4\pi}{r} + \frac{2qz^2}{r^2} \right] \]

Now do H.W.
Prob 3.30 a

Careful does not include this!
\[ E_\text{dipole} = \ ? \]

\[ E_\phi (\rho, \theta) = -\nabla V_\text{dipole} (\rho, \theta) \]

\[ V_\text{dipole} (\rho, \theta) = k \frac{\hat{r} \cdot \vec{p}}{\rho^2} = k \frac{p \cos \theta}{\rho^2} \]

\[ \vec{E}_\phi (\rho, \theta) = \vec{E}_r \hat{r} + E_\theta \hat{\theta} + E_\phi \hat{\phi} \]

\[ E_r = \frac{-\partial V}{\partial r} = k \frac{2p \cos \theta}{\rho^3} \]

\[ E_\theta = -\frac{1}{\rho} \frac{\partial V}{\partial \theta} = k \frac{p \sin \theta}{\rho^3} \]

\[ E_\phi = -\frac{1}{\rho \sin \theta} \frac{\partial V}{\partial \phi} = 0 \quad \implies \quad \text{no } S(\phi) \]

\[ E_\text{dipole} (\rho, \theta) = k \frac{p}{\rho^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\phi} \right) \]

Azimuthal symmetry
\[ E_{\text{dipole}}(r, \theta) = \frac{K P}{r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) \]
Now we solved

$$E(z) \begin{cases} \frac{1}{z^3} & \text{if } z \neq 0 \\ \frac{1}{z^2} & \text{if } z = 0 \end{cases}$$

So is all coal.

So set

$$\mathbf{E}(z) \propto \frac{1}{z^3} \quad \text{in } y \text{ dir.}$$

So here \( \mathbf{p} = \text{polar} = \frac{\pi}{2} \)

\( \frac{1}{z} \mathbf{d} = \text{angle表彰} = \frac{\pi}{2} \)

So have \( \mathbf{E}_{\text{dipole}}(y, \frac{\pi}{2}) = \frac{k \mathbf{P}}{y^3} \left( z \cos \left( \frac{\pi}{2} \right) \mathbf{\hat{r}} + \sin \left( \frac{\pi}{2} \right) \mathbf{\hat{z}} \right) \)

\( = \frac{k \mathbf{P}}{y^3} \left( 0 \mathbf{\hat{r}} + \sin \left( \frac{\pi}{2} \right) \mathbf{\hat{z}} \right) = -\frac{z \mathbf{\hat{z}}}{y^3} \)

\( \mathbf{E} = \frac{k \varepsilon \mathbf{\hat{z}}}{y^3} - \frac{z \mathbf{\hat{z}}}{y^3} \)

\( \theta = \frac{\pi}{2} \)

\( \theta = -\frac{\pi}{2} \)