Our Brute-Force method to solve for $E$...

$$E_p = k \sum_{all\ space} \frac{q}{|r|^2} dy'$$

where $q(y')$ = charge density of source be $4 \pi \delta \delta_s$

$$\frac{1}{|r|} = \frac{\mathbf{p}}{\text{field}} - \frac{\mathbf{p}}{\text{source}}$$

Clearly $\mathbf{p} = \text{const}$ in most case $\Rightarrow$ messy work

In the case of symmetry, there is a nice trick called Gauss's law

so = our 1st "Bag of tricks" as Gauss says it
Gauss's Law 2 way, II) Bit more physical

D) Math

\[ \mathbf{E}(\mathbf{r}) = k \int_{\text{space}} \frac{\mathbf{r}' \times \mathbf{S}(\mathbf{r}')}{|r'^2|} \, dV' \]

Actually, consider:

\[ \nabla \cdot \mathbf{E} = k \int_{\text{space}} \left( \frac{\nabla \cdot \mathbf{J}}{|r'^2|} \right) \, dV' \]

Assume: \( \mathbf{E} = \mathbf{0} \) at origin.

So what is this?

Consider: \( \mathbf{J} = q \delta \) some, which is what you need for \( \mathbf{E}(x',y',z') \) @ a pt.

\[ \nabla \cdot \frac{1}{r^2} \mathbf{\nabla} \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) \]

\[ = \frac{1}{r^2} \frac{\partial}{\partial r} (x' v) = 0 \]
Here is where 2 methods stem from

\[ \oint \mathbf{E} \cdot d\mathbf{A} = \iint_{\partial V} \nabla \cdot \mathbf{E} \, dV \]

\( \nabla \cdot \mathbf{E} \) # 'sources' of \( \mathbf{E} \) that diverge

\[ \text{Flux of } \mathbf{E}, \text{ How many } \mathbf{E} \text{ field lines Slow out of Surface } S \text{ surrounding volume}', \text{ is proportional to The # of sources.} \]

Now both are valid to find out how many sources.

a) Seems to get right to the 'Heart' of what the sources are of \( \mathbf{E} \). BUT will see that we are led to some 'problems'

b) Seems to avoid the 'deep' ?'s about what a 0-D pt source is. BUT the good news is that it doesn't care and lets you get on & work w/ these pt sources!
so div = 0 is true every but ? about \( \frac{d}{r} \) where

That \( \nabla \cdot \left( \frac{e^r}{r^2} \right) = 0 \) both some cause it says pt source of 1 change should not have a divergence...

ie it should not be a pt source!

The problem is \( \nabla \cdot \frac{1}{r^2} = 0 \) math is not wrong it says \( \frac{1}{r} \# 0, \frac{\nabla r}{r} = 0 \) but \( \frac{e^n}{r^2} \) is undefined.

So we have to figure some other way to do this:

Recall \( \int \nabla \cdot \mathbf{E} \, dA = \int \oint \mathbf{E} \cdot d\mathbf{A} \)

so

\( \int \nabla \cdot \left( \frac{e^r}{r^2} \right) \, dA = \int \oint \left( \frac{1}{r} \right) \left( \frac{1}{r^2} \right) \, d\mathbf{A} \)

Imagine the position! Have surf enclose 1 pt. Note neither the surf or the source are specified except that enclose source.

Say take sphere of rad \( R \) around pt change

\( = \oint \int \frac{1}{R^2} \sin \theta d\phi d\theta \)

\( = \frac{4\pi}{R^2} \)

\( = 1 \)
\[ = \left( \int_0^\infty \sin \sigma \, d\sigma \right) \left( \int_0^{2\pi} d\theta \right) = 4\pi \]

Yikes!

\[ \int_{\text{Surf}} - (\frac{\rho}{t^2}) \, d\mathbf{A} = \int_{\text{Surf}} (\frac{\rho}{t^2}) \cdot d\mathbf{A} = 4\pi \]

\[ \nabla \cdot \left( \frac{\rho}{t^2} \right) = 0 \quad \text{everywhere} \]

\[ \text{and undefined at } t = 0 \]

Let \( R \to 0 \) very slowly, yet still get \( 4\pi \).

So what is the size of the sphere around pt charge, despite \( \text{ind} = \text{size} \)?

Failure of Classical to give good picture of pt source of charge! *Math works* (ie: Dirac Delta function) But picture fails. Need QFT.

This is wild! What is the size of an \( \epsilon_0 \) is.

No matter the size, you get \( 4\pi \)?
So \( \nabla \cdot \frac{1}{|\mathbf{r}|^2} = \text{Bizarre!} \)

Because
\[ \nabla \cdot (\mathbf{C}) = 0 \text{ everywhere} \]
\[ \mathbf{v} + \frac{\mathbf{F}}{|\mathbf{F}|} = 0 \]
and
\[ \int \nabla \cdot (\mathbf{C}) \, d\mathbf{r} = \mathbf{S} \cdot \nabla \mathbf{v} \]
\[ = 4 \pi \]
no matter the radius

But since all the problem is \( \mathbf{C} |\mathbf{F}| = 0 \)
\( \frac{1}{|\mathbf{F}|} \) is to take \( \nabla \cdot (\mathbf{v} + \mathbf{F}) \) it is undefined
\[ \mathbf{C} |\mathbf{F}| = 0 \text{ (i.e. } \frac{1}{|\mathbf{F}|} \text{ undefined)} \]

Then the problem is not math!

so Bizarre physically

\[ \text{it as is } \mathbf{C} |\mathbf{F}| = 0 \text{ There is a `source`! of amount } 4\pi \]
\[ \int \nabla \cdot \left( \frac{\mathbf{v}}{1 + r^2} \right) \, dV = \int \frac{\mathbf{v}}{1 + r^2} \cdot \, dA^2 = 4\pi \]

so idea is find a mathematical function that can do this to replace

And the answer is, Dirac delta function: \( \delta(x) = 1 \) D
\( \delta(r) = 3 \) D

So, here how it works

\[ \delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \]

and

\[ \int_{-\infty}^{+\infty} \delta(x') \, dx' = 1 \]

Think about this... The integral is an "area under the curve"

No ordinary function (or physical thing) can do this
It must be \( \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \)

Why not? we have \( \lim_{\delta \to 0} \delta x = dx \)

\( \delta(x) \) itself not legit. But its integral is so delta function always intended to be under an integral.

Now the Dirac Delta function has some properties.

Notice \( \int_{-\infty}^{+\infty} \delta(x) \delta(x) \, dx = \delta(0) \) cause \( \delta(0) = 0 \)

For all \( x \) except \( x = 0 \)

it "picks out" \( \delta(0) \)

Try this \( \delta(a-x) \) instead of \( \delta(x) \)

\( \delta(a-x) \equiv \begin{cases} \infty, & x = 0 \\ 0, & \text{otherwise} \end{cases} \)

Then \( \int_{-\infty}^{+\infty} \delta(a-x) \, dx = 1 \)

And \( \int_{-\infty}^{+\infty} \delta(x) \delta(a-x) \, dx = \delta(a) \leq \text{cool!} \)
Note, the integral $\int_{-\infty}^{+\infty} \gamma \, dx$ need not $\gamma$ only 
has to go $\int_{a-\epsilon}^{a+\epsilon} \gamma (a-x) \, dx = 1$

is $\epsilon \to 0$, or at least approach it?

Example: $1.14$ 
$\int_{0}^{3} x^3 \gamma (x-2) \, dx = \gamma(2) = 2^{3} = 8$

It is not 
$\int_{0}^{3} x^3 \, dx$

Homework: 
1.43 a-d
1.45 a only!
New 3-D $S_3^3$ 

\[ S_3^3 = S(x)S(y)S(z) = \begin{cases} 0 & \text{everywhere except} \\ \infty & \text{at } x = y = z = 0 \end{cases} \]

\[ \text{Then} \]

\[ \int_{\text{all space}} S_3^3 \, dp = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S(x)S(y)S(z) \, dx \, dy \, dz = 1 \]

\[ \int_{\text{all space}} S_3^3 \, dp = S(S_3^3 - \lambda) \]

and \[ \int_{\text{all space}} S_3^3 \, dp = S(S_3^3 - \lambda) \]

\[ \text{picks out} \]
\[ \int_{\mathcal{V}} \nabla \cdot \left( \frac{\mathbf{F}}{r^2} \right) \, d\mathbf{V} = \int_{\partial \mathcal{V}} \mathbf{F} \cdot \mathbf{n} \, d\mathbf{A} = 4\pi \]

So the idea is find a mathematical function that can do this to replace

So if we replace \( \nabla \cdot \left( \frac{\mathbf{F}}{r^2} \right) \) with \( 4\pi S^2(\mathcal{F}) \)

we get

\[ \int_{\mathcal{V}} 4\pi S^2(\mathcal{F}) \, d\mathbf{V} = (4\pi)(1) \]

YEAH!

More generally, where \( \mathcal{F} = \mathbf{F}^{\text{field}} - \mathbf{F}^{\text{source}} \)

\[ \nabla \cdot \left( \frac{\mathbf{F}}{r^2} \right) = 4\pi S^2(\mathcal{F}) \]
OK, \( \nabla \cdot \left( \frac{\vec{F}}{12\pi} \right) = 4\pi S^2 \left( \vec{F} \right) \) --- got it.

Now back to our 1D math derivation of Gauss's Law:

Brute force calc of \( \vec{E} \)

\[
\vec{E}(\vec{r}) = k \sum_{\text{all space}} \frac{\vec{r}}{|\vec{r}|^2} \cdot 3 \left( \vec{F}'' \right) \, d\vec{r}',
\]

New take \( \nabla \cdot \) of both sides:

\[
\nabla \cdot \vec{E}(\vec{r}) = k \sum_{\text{all space}} \nabla \cdot \left( \frac{\vec{r}}{|\vec{r}|^2} \right) \cdot 3 \left( \vec{F}'' \right) \, d\vec{r}',
\]

\[
= k \sum_{\text{all space}} 4\pi S^2 \left( \vec{F}'' \right) \cdot 3 \left( \vec{F}'' \right) \, d\vec{r}',
\]

\[
= k \sum_{\text{all space}} 4\pi S^2 \left( \vec{F}'' \right) \cdot 3 \left( \vec{F}'' \right) \, d\vec{r}',
\]

\[
= k \sum_{\text{all space}} 4\pi S^2 \left( \vec{F}'' \right) \cdot 3 \left( \vec{F}'' \right) \, d\vec{r}',
\]

\[
\nabla \cdot \vec{E} = \frac{1}{4\pi \varepsilon_0} \cdot 4\pi S^2 \left( \vec{F}'' \right) = \frac{8C^2}{3}
\]
wow! \[ \nabla \cdot \mathbf{E} = \pm \frac{1}{\varepsilon_0} \mathbf{S}(\mathbf{r}) \] where \( \mathbf{S}(\mathbf{r}) = \) charge density!

Differential form of Gauss's Law!

Now

\[ \nabla \cdot \mathbf{E} = \pm \int \frac{\mathbf{S}(\mathbf{r})}{\varepsilon_0} \mathrm{d}V = \int \mathbf{E} \cdot \mathrm{d}A \]

Integral form of Gauss's Law!

\[ \frac{Q_{\text{total}}}{\varepsilon_0} = \int \mathbf{E} \cdot \mathrm{d}A \]

Huge!

\[ \frac{\text{Flux of E-field Lines}}{\varepsilon_0} \] more than proportional to \( Q_{\text{total}} \) is equal.
Now Gauss's Law By Physical Argument!

\[ \int \mathbf{E} \cdot d\mathbf{A} = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \]

got there by math, now by more physical idea

Field lines

Stream lines

\[ |\mathbf{E}| \propto \text{length} \]

\[ \frac{|\mathbf{E}|}{|V|} \propto \text{length} \]

\[ \frac{1}{V^2} \propto \text{density of lines} \]

\[ \frac{1}{V^2} \propto \text{density} \]

3-D works

ie density \( \propto \frac{1}{r^2} \) for sphere

OK
So Draw $E$ fields!

*rules: 1st of # of lines $\Rightarrow$ up to you but most 'scale'. It $+2e$ has 4
  then $6e$ must have 12
  & so on*

2. Field lines
   Start on $(\pm)$ charges
   End on $(\pm)$ charges

3. Field lines $\neq$ cross
   Would mean $\Rightarrow$ this
   $\Rightarrow$ Both $\Rightarrow$
   & same time
   $\Rightarrow$ impossible!

4. Can't terminate in mid-air but
   can extend to $\infty$

   Ex. 

   Start to draw from each charge
   Then
   (a) connect them
   (b) repel them or
   (c) extend to $\infty$
Now consider idea of flux:

\[ \Phi = \text{Flux of vector field } \mathbf{V} \]

\[ \Phi = \int \int_\Sigma \mathbf{E} \cdot d\mathbf{A} \]

Clearly is:

Drawing vector field
stream lines, where

\[ \mathbf{v} \propto \text{line density} \]

\[ \mathbf{E} \cdot d\mathbf{A} = \left( \frac{\# \text{lines}}{\text{Area}} \right) \text{Area} = \oint \text{# lines passing through } d\mathbf{A} \]
So \( \mathbf{E} \cdot d\mathbf{A} = d (\# \text{ lines passing thru } dA) \)

So \( \oint \mathbf{E} \cdot d\mathbf{A} = \# \text{ lines passing thru Area surrounding } \nabla \text{ sources of } \mathbf{E} \)

\[ \begin{array}{c}
\# \text{ of lines outside }
\text{contribute}
\# \text{ of lines}
\text{due entirely}
\text{to}
\text{enclosed}.
\end{array} \]

\[ \begin{array}{c}
\# \text{ of lines to the flux cause in } = (-)\text{ line out} = (+)\text{ line out}
0
\end{array} \]

OK got it. Consider 1 pt charge at the origin. The flux of \( \mathbf{E} \) thru sphere of radius \( r \) is...

\( \oint \mathbf{E} \cdot d\mathbf{A} = k \oint (\frac{q}{4\pi \epsilon_0 r^2}) \cdot (r^2 \sin \theta d\theta d\phi) \hat{r} = kE_0 4\pi \)

\[ \begin{array}{c}
\frac{1}{4\pi \epsilon_0} \cdot 4\pi = \frac{1}{\epsilon_0}
\end{array} \]

So \( \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\epsilon_0} \)

The flux of \( \mathbf{E} \) due to pt source of charge.
Now suppose there are many charges distributed throughout space (\( \mathbf{e} \)).

Each would contribute \( \mathbf{E}_o \) so

\[
\int \mathbf{E}_o \cdot d\mathbf{A} = \sum_{i=1}^{n} \left( \int \mathbf{E}_o \cdot d\mathbf{A} \right) = \sum_{i=1}^{n} \left( \frac{1}{\varepsilon_0} \right)
\]

\[
\int \mathbf{E}_o \cdot d\mathbf{A} = \frac{Q_{\text{total enclosed}}}{\varepsilon_0}
\]

---------------

**DONE!**

Gauss's Law!

\[
\int (\nabla \cdot \mathbf{E}) \, d\mathbf{V} = \int \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{total enclosed by } S}}{\varepsilon_0}
\]

S in differential form = Max's 1\textsuperscript{st} Equation

\[
\nabla \cdot \mathbf{E} = \frac{\mathbf{S}}{\varepsilon_0} \quad \mathbf{S} = \text{change density.}
\]
Gauss's law...

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

How is it useful?

by example.

Say

$$\mathbf{E}(\mathbf{p}) = ?$$

Note before just looking @ $\mathbf{E}(\mathbf{0})$ also asking

a more general?

Brute Force: $d\mathbf{E}(\mathbf{p}) = k \frac{e^2}{r^2} \hat{r} = k \sigma \frac{\mathbf{r} \times \hat{r}}{|\mathbf{r}|^3}$

Now! Choose Gaussian....

1) Sphere Cause of Symmetry
2) choose $r > R$
Gaussian Sphere of radius \( R \)

Since

\[
\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{tot}} \text{-enclosed}}{\varepsilon_0}
\]

easy

\[
= \frac{\sigma \times 4\pi R^2}{\varepsilon_0} = \left( \frac{\text{charge}}{\text{Area}} \right) \frac{1}{\varepsilon_0}
\]

\[\mathbf{E}(\mathbf{r}) = |\mathbf{E}_0| \mathbf{r} \]

\[
d\mathbf{A} = 1d\mathbf{A}
\]

\[
\oint_S |\mathbf{E}_0| d\mathbf{A} \cdot \mathbf{r} \cdot r = 1
\]

\[
\mathbf{E}(\mathbf{r}) \oint_S d\mathbf{A} 
\]

\[
\oint_S d\mathbf{A} \cdot \mathbf{E}(\mathbf{r}) = 1
\]

\[
\oint_S d\mathbf{A} = 4\pi r^2
\]

\[
\mathbf{E}(\mathbf{r}) = \frac{\sigma \times 4\pi R^2}{\varepsilon_0} = \frac{Q_{\text{tot}}}{4\pi \varepsilon_0 R^2}
\]

\[
E_{\text{cr}} = \frac{\sigma \times 4\pi R^2}{4\pi \varepsilon_0 R^2} = \frac{1}{4\pi \varepsilon_0} \frac{Q_{\text{tot}}}{r^2} = \frac{Q_{\text{tot}}}{4\pi \varepsilon_0 r^2}
\]

\[
E_{\text{cr}} = \frac{\sigma \times 4\pi R^2}{\varepsilon_0}
\]
...it as is \( \sqrt{-1} = i \) there is a 'source'
of amount \( \delta \).

OK.

1.) Mathematically we can send
a function that does just that
The Dirac Delta Function!

Here it is:

\[
\begin{align*}
1-D & \quad \delta(x) = \begin{cases} 
0 & \text{is } x \neq 0 \\
\infty & \text{is } x = 1
\end{cases} \\
3-D & \quad \delta(r) = \begin{cases} 
0 & \text{is } r \neq 0 \\
\frac{1}{r^2} & \text{is } r = 0
\end{cases}
\end{align*}
\]

Then
\[
\int_{-\infty}^{+\infty} \delta(x') \, dx' = 1
\]

\[
\frac{1}{2} \int_{-\infty}^{+\infty} \delta(x) \, dx \cdot \int_{-\infty}^{+\infty} \delta(x') \, dx' = \delta(x) \int_{-\infty}^{+\infty} \delta(x) \, dx = \delta(0)
\]

"It picks off" \( \delta(x) \)
So back to our problem

\[ \nabla \cdot \left( \frac{\varphi}{l^2} \right) = 4\pi \delta(r) \]

so \[ \nabla \cdot \vec{E} = k \sum \frac{\delta^3(r-r_j)}{4\pi r} \delta(x, y, z) \]

or

\[ \int_V \nabla \cdot \vec{E} \, dV = \sum \frac{\delta \delta^3(r-r_j)}{4\pi r} \delta(x, y, z) \]