E.F. Deveney/BSC Physics PH438 Gauss's Law

Note Title 10/21/2004

Our Brute-Force method to solve for $\vec{E}$ ...

$$\vec{E}(\vec{r}) = k \int_{\text{all space}} \frac{\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2} \rho(\vec{r}\,')\, d\tau'$$

$\vec{r}$ — field position

where $\rho(\vec{r}\,')$ = charge density of source be it, λ, σ, ρ

& $\vec{\mathscr{r}} \equiv \vec{r}_{\text{field posit}} - \vec{r}_{\text{source}}$

Clearly $\hat{\mathscr{r}} \neq$ const in most case $\Rightarrow$'s messy work

In the case of symmetry, there is a Nice trick called gauss's law

so = our 1st "Bag of tricks" as Gauss says to

Gauss's Law 2 way. 1) strickly math 2) Bit more physical

see Next page 1st

1) Math

$$\vec{E}(\vec{r}) = k \int_{\text{space}} \frac{\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2} \rho(\vec{r}')\, d\tau'$$

$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

take $\vec{\nabla}\cdot\vec{E}$

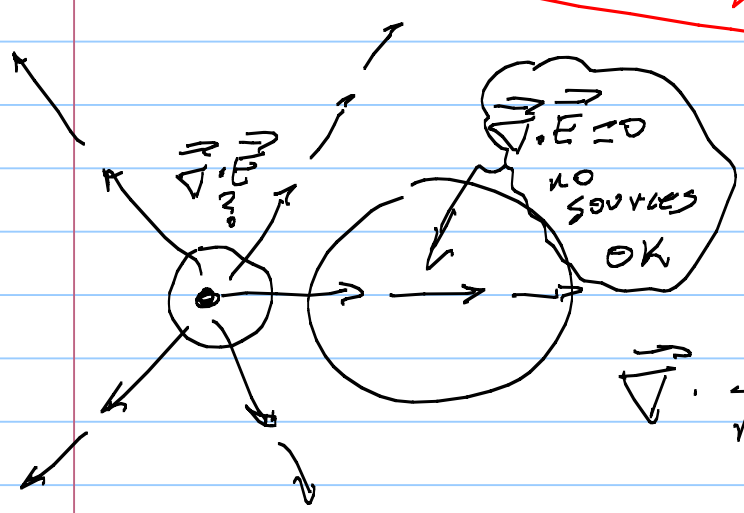
Actually, consider a pt locat @ origin.

$$\vec{\nabla}\cdot\vec{E} = k \int \left(\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2}\right) \underbrace{\rho(\vec{r}')\, d\tau'}_{\text{scalar}}$$

Assume: pt = 0 Dimensions

So what is this?

Consider: $\vec{r}' = 0$ = pt source which is what you need for $\rho(x', y', z')$ @ a pt



so

$$\vec{\nabla}\cdot\frac{1}{r^2}\hat{r}$$

you already did for H.W.

$$\nabla_{\text{sphr}}\cdot\frac{1}{r^2}\hat{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\frac{1}{\cancel{r^2}}\right) = 0$$

Here is where 2 methods stem from

$$\underbrace{\int_{V} (\vec{\nabla}\cdot\vec{E})\,d\tau}_{\text{(a)}} = \underbrace{\oint_{S} \vec{E}\cdot d\vec{A}}_{\text{(b)}} \quad \left\{ \begin{array}{l} \text{Gauss's} \\ \text{Green's} \\ \text{Div} \end{array} \right. \text{Theorem}$$

$\vec{\nabla}\cdot\vec{E} \propto \dfrac{\#\text{ 'sources' of } \vec{E} \text{ that diverge}}{\text{Volume}}$

Flux of $\vec{E}$, How many $\vec{E}$ field lines flow out of surface 'S' surrounding volume' is proportional to the # of sources.

Now Both are Valid to find out how many sources.

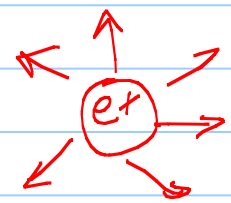
a.) Seems to get right to the 'Heart' of what the sources are of $\vec{E}$ BUT will see that we are led to some 'problems'

b.) Seems to avoid the 'deep' ?'s about what a 0-D pt source is BUT the good news is that it doesn't care and lets you get on & work w/ these pt sources!

So div = 0 is true everywhere but ? about $|\vec{r}|=0$

That $\vec{\nabla}\cdot\left(\frac{\hat{r}}{|\vec{r}|^2}\right)=0$ bothersome cause it says pt source of 1 charge should not have a divergence ... ie it should not be a pt source!

zero dimension pt source



The problem is @ $|\vec{r}|=0$

math is not wrong, it says $|r|\neq 0$, $\vec{\nabla}\cdot\vec{v}=0$ BUT @ $|r|=0$, $\vec{\nabla}\cdot\vec{v}$ is undefined!

So we have to figure some other way to do this:

recall $$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{E}\,d\mathcal{V} = \int_{\substack{\text{surf}\\ \text{of } \mathcal{V}}} \vec{E}\cdot d\vec{A}$$

so

$$\int_{\mathcal{V}} \vec{\nabla}\cdot\left(\frac{\hat{r}}{|\vec{r}|^2}\right)d\mathcal{V} = \int_{\text{surf}} \left(\frac{1}{|\vec{r}|^2}\hat{r}\right)\cdot d\vec{A}$$

imagine the positron! Have surf enclose it! Note neither the $\mathcal{V}$ or the surf specified except that enclose source.

say take sphere of rad R around pt charge

$$= \int_{\text{surf}} \left(\frac{1}{R^2}\hat{r}\right) R^2 \sin\theta\, d\phi\, \hat{r}$$

$\hat{r}\cdot\hat{r}=1$

$$= \left(\int_0^{\pi} \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\varphi\right) = 4\pi$$

$$2 \qquad 2\pi$$

Yikes!

$$\int_{V} \vec{\nabla}\cdot\left(\frac{\hat{r}}{|\vec{r}|^2}\right) dV = \int_{\substack{\text{Surf}\\ V}} \left(\frac{\hat{r}}{|\vec{r}|^2}\right)\cdot d\vec{A} = 4\pi$$

independent of R = size of sphere around pt charge & despite

Let R → ∞ & still get 4π

So what is the size, r, of the source?

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{|\vec{r}|^2}\right) = 0 \text{ everywhere}$$

and undefined @ $\vec{r} = 0$

Failure of Classical to give good Picture of pt source of charge! * Math works (ie Dirac Delta Funct) But picture fails. Need Q.T.

This is wild!

What is the size of an e^-, is

No matter the size, you get 4π?

So $\vec{\nabla} \cdot \frac{\hat{r}}{|\vec{r}|^2}$ = Bizarre!

Because $\vec{\nabla} \cdot (\ ) = 0$ everywhere but $|\vec{r}| = 0$?

and

$$\int \vec{\nabla} \cdot (\ )\, d\tau = \int (\ ) \cdot d\vec{A}$$

$$= 4\pi$$

no matter the radius

But since all the problem is @ $|\vec{r}| = 0$

& if take $\vec{\nabla} \cdot (\ )$ it is undefined

@ $|\vec{r}| = 0$ (ie $\frac{1}{|r|}$ = undefined)

Then the problem is not math!

So Bizarre physically

it as if @ $|\vec{r}| = 0$, there is a 'source' of amount 4π

$$\int_{\mathcal{V}} \vec{\nabla} \cdot \left(\frac{\hat{r}}{|\vec{r}|^2} \right) d\mathcal{V} = \int_{\substack{\text{surface} \\ \mathcal{V}}} \left(\frac{\hat{r}}{|\vec{r}|^2} \right) \cdot d\vec{A} = 4\pi$$

so idea is find a mathematical funct that can do this to replace

And the answer is, Dirac delta funct! $\delta(x) = 1\text{-D}$

$\delta^3(\vec{r}) = 3\text{-D}$

So, here how it works

$$\delta(x) \equiv \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

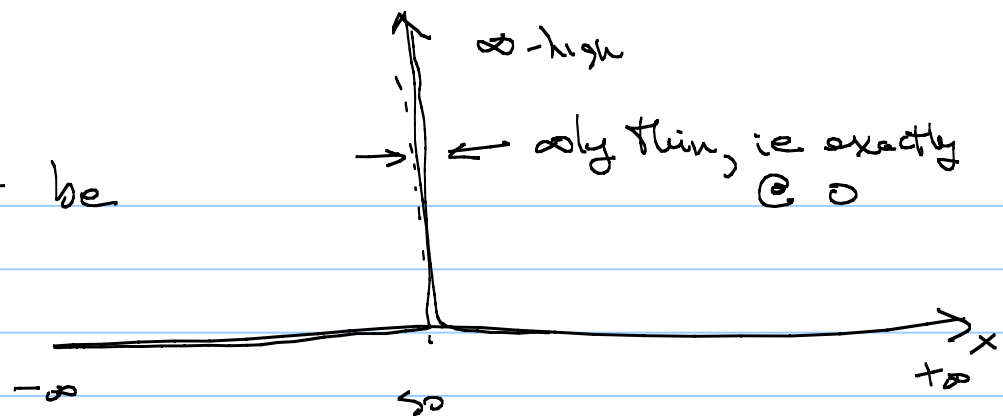
and

$$\int_{-\infty}^{+\infty} \delta(x')\, dx' = 1$$

Thunk about this..... The integral is an "area under the curve"

No ordinary function (or physical thing) can do this

it must be



so that area = 1

why not? we have $\lim_{\Delta x \to 0} \Delta x = dx$

so and that's 'OK' mathematically.

Gripe: $\delta(x)$ itself not legit funct But its integral is so delta function always intended to be under an integral

Now the Dirac Delta funct has fun properties

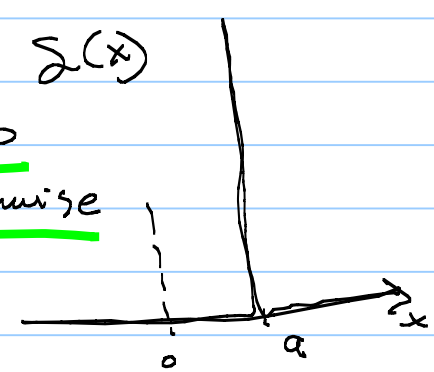
Notice $$\int_{-\infty}^{+\infty} f(x')\,\delta(0)\,dx' = f(0)$$ cause $\delta(0) = 0$ for all x except $x=0$

it "picks out" $f(0)$

Try this $\delta(a-x)$ instead of $\delta(x)$

$$\delta(a-x) \equiv \begin{cases} \infty, & x=0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore $\int_{-\infty}^{+\infty} \delta(a-x')\,dx' = 1$ =



and $\int_{-\infty}^{+\infty} f(x')\,\delta(a-x')\,dx' = f(a)$ ⟸ cool!

Note, the integral $\int_{-\infty}^{+\infty}$ need not & only

has to go $\int_{a-\epsilon}^{+\epsilon+a} \delta(a-x')\,dx' = 1$ ie 'contain' the delta

& $\epsilon \to 0$, or @ least approach it?

Ex: 1.14 $\int_0^3 x^3 \delta(x-2)\,dx = f(2) = 2^3 = 8$

it is not $\int_0^3 x^3 \Big|_2 dx$

H.W. Griffs
1.43 a–d
1.45 a only!

Now 3-D $\delta^3(\vec{r})$

$$\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z) \equiv \begin{cases} 0 & \text{everywhere except} \\ \infty & @\ x=y=z=0 \end{cases}$$

so that

$$\int_{\substack{\text{all} \\ \text{space} \\ \forall}} \delta^3(\vec{r})\,d\tau = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1$$

and

$$\int_{\substack{\text{all space} \\ \forall}} f(\vec{r})\,\underbrace{\delta^3(\vec{r}-\vec{a})}_{\text{picks out}}\,d\tau = f(\vec{a})$$

$$\int_{\mathcal{V}} \vec{\nabla}\cdot\left(\frac{\hat{r}}{|\vec{r}|^2}\right) d\mathcal{V} = \int_{\substack{\text{surface}\\ \mathcal{V}}} \left(\frac{\hat{r}}{|\vec{r}|^2}\right)\cdot d\vec{A} = 4\pi$$

so idea is find a mathematical funct that can do this to replace

So if we 'replace' $\vec{\nabla}\cdot\left(\frac{\hat{r}}{|\vec{r}|^2}\right)$ w/ $4\pi\,\delta^3(\vec{r})$

we get

$$\int_{\mathcal{V}} 4\pi\,\delta^3(\vec{r})\, d\mathcal{V} = (4\pi)(1)$$

YEAH

More generally, where $\vec{\Omega} \equiv \vec{r}_{field} - \vec{r}_{source}$

$$\vec{\nabla}\cdot\left(\frac{\hat{\Omega}}{|\vec{\Omega}|^2}\right) = 4\pi\,\delta^3(\vec{r})$$

OK; $\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2}\right) = 4\pi\,\delta^3(\vec{r})$ ---- got it.

Now Back to our 1.) Math derivation of Gauss's Law

Brute Force calc of $\vec{E}$

$$\vec{E}(\vec{r}) = K\int_{\substack{\text{all}\\\text{space}}} \frac{\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2}\,\rho(\vec{r}\,')\,d\tau'$$

Now take $\vec{\nabla}\cdot$ of both sides

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = K\int_{\forall} \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2}\right)\rho(\vec{r}\,')\,d\tau'$$

$$= K\int_{\forall} 4\pi\,\delta^3(\vec{\mathscr{r}})\,\rho(\vec{r}\,')\,d\tau'$$

$$= K\int_{\forall} 4\pi\,\underbrace{\delta^3(\vec{r}-\vec{r}\,')}_{\equiv\begin{cases}\infty & \vec{r}\,'=\vec{r}\\ 0 & \text{otherwise}\end{cases}}\,\rho(\vec{r}\,')\,d\tau'$$

& $\int f(\vec{r}\,')\,\delta^3(\vec{r}-\vec{r}\,')\,d\tau' = f(\vec{r})$

$$= K\,4\pi\,\rho(\vec{r})$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}4\pi\,\rho(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$$

Wow! $\underbrace{\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho(\vec{r})}_{\text{differential form of Gauss's Law!}}$ where $\rho(\vec{r})$ = charge density!

Now

$$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{E}\, d\mathcal{V}' = \underbrace{\frac{1}{\varepsilon_0}\int_{\mathcal{V}} \rho(\vec{r})\, d\mathcal{V}'}_{\underbrace{\int \rho(r)\, d\mathcal{V}'}_{=Q_{TOT}}} = \int_{S \text{ of } \mathcal{V}} \vec{E}\cdot d\vec{A}$$

$$\boxed{\frac{Q_{TOT}}{\varepsilon_0} = \int \vec{E}\cdot d\vec{A}}$$

integral form of Gauss's Law!

means....

HUGE!

$$\underbrace{\int_S \vec{E}\cdot d\vec{A}}_{\text{Flux of E-field Lines}} = \frac{Q_{\text{total}}}{\varepsilon_0} \text{ (enclosed by surface)}$$

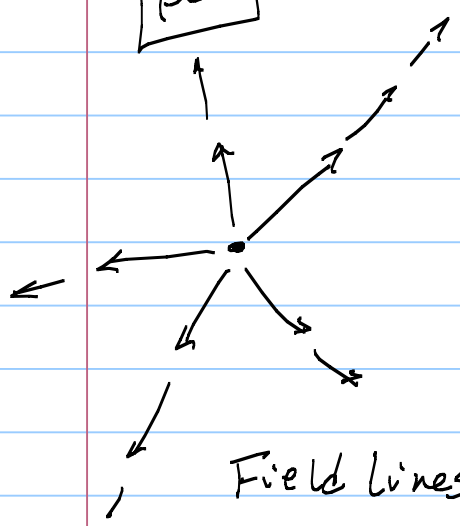
more than proportional now is EQUAL

Now Gauss's Law By Physical Argument!

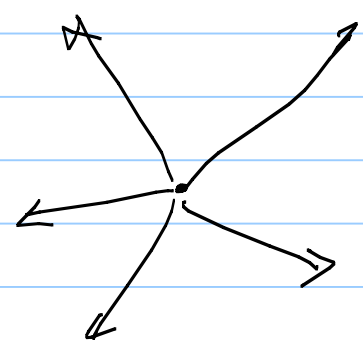
$$\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{E}\, d\tau = \int_{S} \vec{E}\cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}$$

got there by math, now by more physical idea

$$\boxed{\frac{\hat{r}}{|\vec{r}|^2}}$$



Field lines



Streamlines

$|\text{mag}| \propto \text{length}$

$|\vec{V}| \propto \text{length}$

$|\text{mag}| \propto$ density of lines

$|\vec{V}| \propto$ density

3-D works

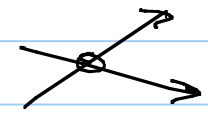
ie density $\propto \frac{1}{\text{Area sphere}} = \frac{1}{r^2}$ OK

So Draw $\vec{E}$ Fields!

rules: 1st of # of lines ⇒ up to you but must 'scale'. If $+2q$ has 4 then $6q$ must have 12 & so on

2. Field lines
   Start on (+) charges
   End on (−) charges

3.) Field lines ≠ cross

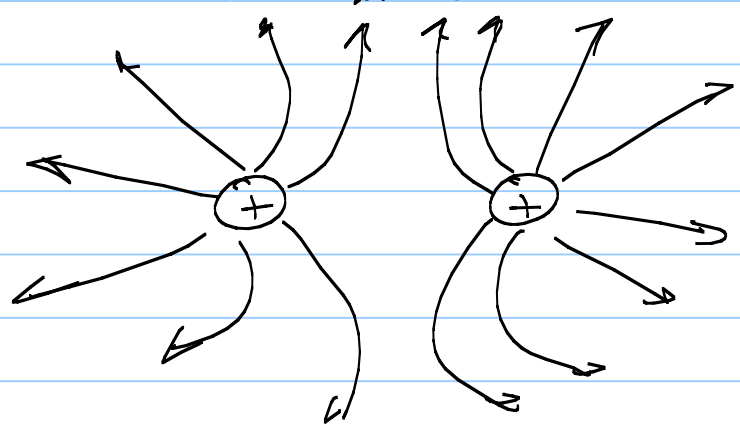


would mean ⊖ this pt

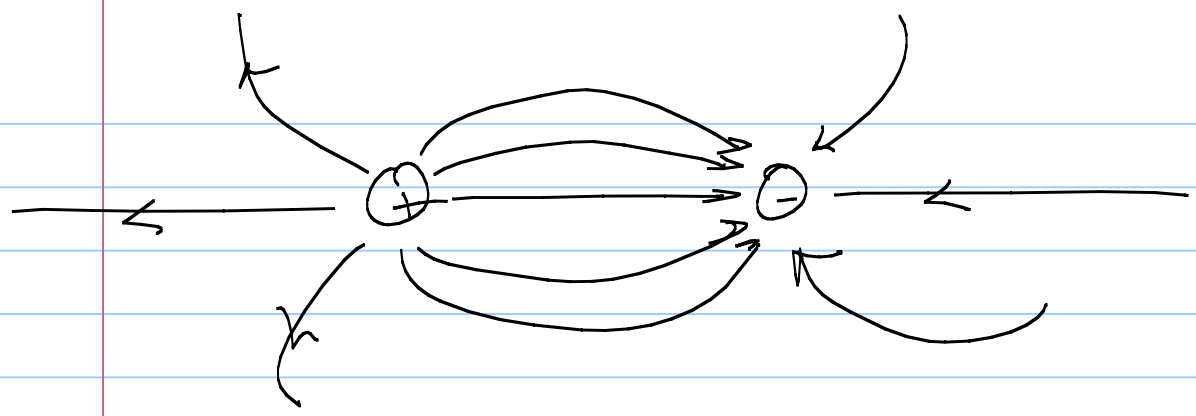
Both ⊖ same time = impossible!

4.) Can't Terminate in mid air but can extend to ∞

ex:



Start to draw from each charge, Then
a) connect them
b) repel them or
c) extend to ∞

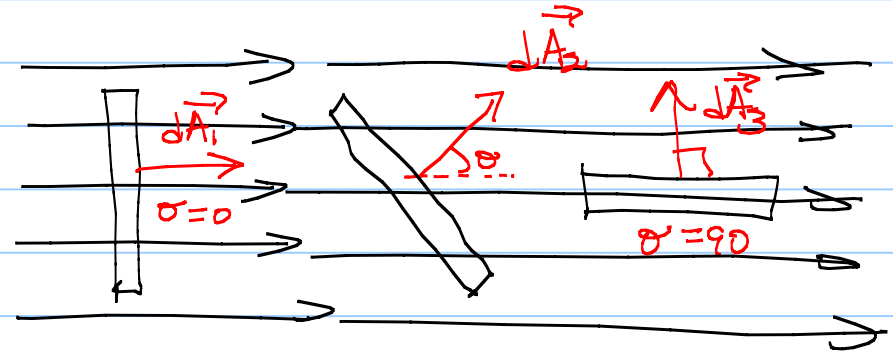


Now Consider idea of flux:

$\Phi_{\vec{V}} \equiv$ Flux of <u>vector field</u> $\vec{V}$ so

$$\Phi_E \equiv \int_S \vec{E} \cdot d\vec{A}$$

clearly is drawing vector field stream lines, where $|\vec{V}| \propto$ line density $\Rightarrow$

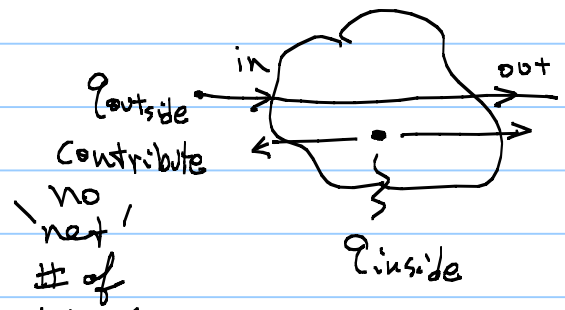


$$\vec{E} \cdot d\vec{A} = \left(\frac{\#\text{Lines}}{\text{Area}}\right)(\text{Area}) = d(\#\text{Lines passing thru } |dA|)$$

So $\vec{E}\cdot d\vec{A} = d\left(\#\text{ Lines passing thru } (dA)\right)$

So

$$\oint_{\text{Surface}} \vec{E}\cdot d\vec{A} = \#\text{ Lines passing thru Area surrounding "Sources" of } \vec{E}$$

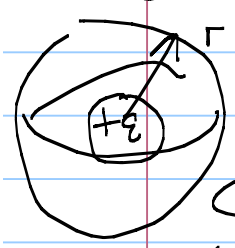


Contribute no 'net' # of Lines to the Flux cause

in = (−) line
out = (+) line
0

# of lines due entirely to Q enclosed!

OK got it. Consider 1 pt charge @ the origin $\vec{r} = \vec{r}$



The Flux of $\vec{E}$ thru sphere of radius r is ...

$$\oint \vec{E}\cdot d\vec{A} = K\oint \left(q\,\frac{\hat{r}}{|\vec{r}|^2}\right)\cdot\left(r^2\sin\theta\, d\theta\, d\phi\right)\hat{r} = Kq\,4\pi$$

$$= \frac{1}{4\pi\varepsilon_0}\, q\, 4\pi = q/\varepsilon_0$$

So $\oint \vec{E}_{\substack{pt\\source}}\cdot d\vec{A} = q/\varepsilon_0$ } The 'Flux' of $\vec{E}$ due to pt source of charge!

Now Suppose there are many charges distributed throughout space ($\mathcal{V}$)

each would contribute $\varepsilon/\varepsilon_0$ so

$$\oint_S \vec{E}\cdot d\vec{A} = \sum_{i=1}^{n}\left(\oint_S \vec{E}_i\cdot d\vec{A}\right) = \sum_{i=1}^{n}\left(\frac{1}{\varepsilon_0}\varepsilon_i\right)$$

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{TOT\ enclosed}}{\varepsilon_0}$$

DONE!

Gauss's Law!

$$\int_{\mathcal{V}} (\vec{\nabla}\cdot\vec{E})\,d\tau = \oint_{\substack{\text{surf}\\ \text{enclosing } \mathcal{V}}} \vec{E}\cdot d\vec{A} = \frac{Q_{\substack{\text{total}\\ \text{enclosed by } S}}}{\varepsilon_0}$$

& in differential form = Maxis 1st Equation

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

ρ = charge density.

Gauss's law ...

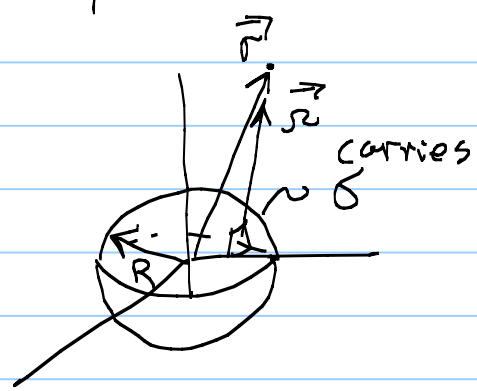
$$\int_{\mathcal{V}} (\vec{\nabla}\cdot\vec{E})\,d\tau = \int_S \vec{E}\cdot d\vec{A} = \frac{Q_{tot,encl}}{\varepsilon_0}$$

How is it useful?

By example.

actually, do this for pt charge @ origin 1st!

Say



$$\vec{E}(\vec{r}) = ?$$

Note before just looked @ $\vec{E}(z)$, here asking a more general?

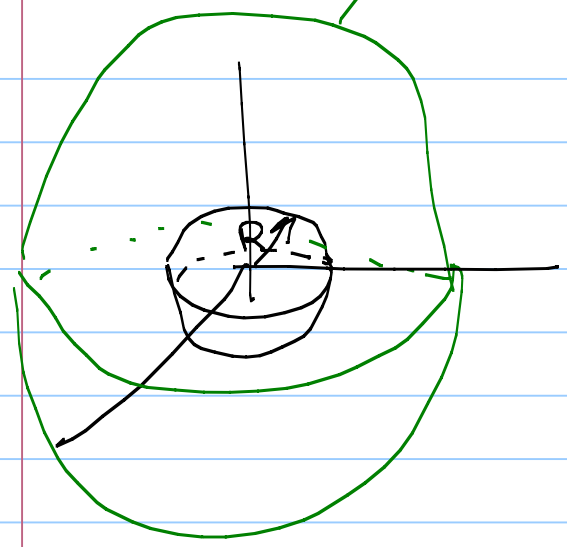
Brute Force: $d\vec{E}(\vec{r}) = K\frac{dq}{|\vec{\mathscr{r}}|^2}\hat{\mathscr{r}} = K\sigma r'^2 \sin\theta\, d\theta\, d\phi \underbrace{\frac{\hat{\mathscr{r}}(\theta,\phi)}{|\vec{\mathscr{r}}(\theta,\phi)|^2}}_{\text{Tough.}}$

Now! Choose Gaussian....

1.) Sphere Cause of symmetry

2.) choose $r > R$

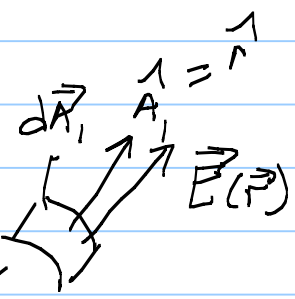
Gaussian sphere of radius r, $r > R$ (ie encloses charges)



Since

$$\int_S \vec{E}\cdot d\vec{A} = \underbrace{\frac{Q_{TOT\text{-}enclosed}}{\varepsilon_0}}_{\text{easy}}$$

$$= \frac{\sigma * 4\pi R^2}{\varepsilon_0} = \frac{\left(\frac{\text{charge}}{\text{Area}}\right)(\text{Area})}{\varepsilon_0}$$

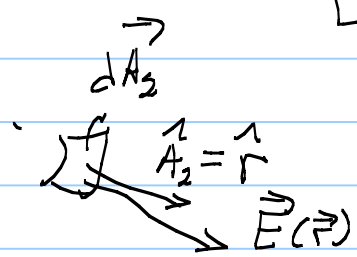


$$\vec{E}(\vec{r}) = |E(r)|\,\hat{r}$$

$$d\vec{A} = |dA|\,\hat{r}$$

$$\int_S |E(r)|\,|dA|\,\underbrace{\hat{r}\cdot\hat{r}}_{=1}$$

Constant @ $|r|$



$$E(r)\underbrace{\int_S dA}_{\text{just } 4\pi r^2}$$

Note r Not R

$$E(r)\,4\pi r^2 = \frac{\sigma 4\pi R^2}{\varepsilon_0}$$

or

$$\vec{E}(r) = \frac{\sigma 4\pi R^2}{4\pi\varepsilon_0 r^2}\,\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{Q_{TOT}}{r^2}\,\hat{r} = K\frac{Q_{TOT}}{r^2}\,\hat{r}$$

. . . .

it as is @ $|\vec{r}|=0$, there is a 'source' of amount 4π

OK,

1.) Mathematically we can find a function that does just that 'the Dirac Delta Function'

here it is

1-D $\quad \delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 1 \end{cases}$

3-D $\quad \delta(\vec{r}) = \begin{cases} 0 & \text{if } \vec{r} \neq 0 \\ 1 & \text{if } \vec{r} = 0 \end{cases}$

Then $\displaystyle\int_{-\infty}^{+\infty} \delta(x')\,dx' = 1$

& $\displaystyle\int_{-\infty}^{+\infty} f(x)\,\delta(x')\,dx' = f(0)\int_{-\infty}^{+\infty} \delta(x')\,dx' = f(0)$

"It picks off" $f(x)$

So Back to our problem

$$\vec{\nabla} \cdot \left(\frac{\hat{\mathscr{r}}}{|\vec{\mathscr{r}}|^2} \right) = 4\pi \delta^3(\vec{\mathscr{r}})$$

So $$\vec{\nabla} \cdot \vec{E} = K \int 4\pi \delta^3(\vec{r} - \vec{r}\,') \rho(\vec{r}\,') d\tau'$$

$$= \frac{1}{\varepsilon_0} \rho(\vec{r})$$

or

$$\int_V \vec{\nabla} \cdot \vec{E} \, d\tau = \oint_S \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} \int_V \rho \, d\tau = \frac{1}{\varepsilon_0} Q_{encl_S}$$