E.F. Deveney/BSC Physics: PH438 Electrostatics 2

Note Title 10/18/2004

All we've been trying to do is solve for $\vec{E}(\vec{r})$ due to some source charge distribution λ, σ, ρ located @ $\vec{r}\,'$ (ex; x', y', z')

by Brute Force

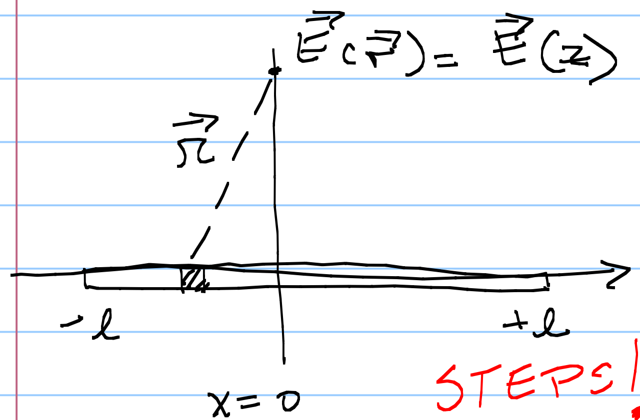
$$\vec{E}(\vec{r}) = \int_{V} \frac{\rho(x', y', z')\, dV'}{|\vec{\mathscr{r}}|^2}\, \hat{\mathscr{r}}(x', y', z')$$

& saw that life gets messy.

So, slow down and

1) Be smart, exploit symms
2) " " ; Gauss' Law
3) tricks
4) $\Longrightarrow$ go to (scalar) $V(x, y, z)$
   then $\vec{E}(\vec{r}) = -\vec{\nabla}(V(\vec{r}))$

Ex: 2.1 in Griff, Find $\vec{E}(\vec{r})$ a distance z above the midpoint of a straight line segment of length 2ℓ & uniform λ.



STEPS!

$|\vec{\mathscr{r}}| = \sqrt{\vec{\mathscr{r}} \cdot \vec{\mathscr{r}}}$

so $|\mathscr{r}|^2 = \vec{\mathscr{r}} \cdot \vec{\mathscr{r}}$

Brute Force: 1.) $dq = \lambda \, dx'$

3.) $\vec{\mathscr{r}} = \vec{r} - \vec{r}_{source}$

$= (0,0,z) - (x,0,0)$

$= -x\hat{i} + 0\hat{j} + z\hat{k}$

2.) $d\vec{E}(z) = \dfrac{k \lambda \, dx'}{|\mathscr{r}|^2} \hat{\mathscr{r}}$

4.) $\hat{\mathscr{r}} = \dfrac{\vec{\mathscr{r}}}{|\vec{\mathscr{r}}|}$

5.) $d\vec{E}(z) = \dfrac{k \lambda \, dx'}{[x'^2 + z^2]^{3/2}} \left(-x\hat{i} + z\hat{k}\right)$

$$\vec{E}(z) = \left(k\lambda \int_{-\ell}^{+\ell} \frac{-x' \, dx'}{[x'^2 + z^2]^{3/2}}\right)\hat{i} + \left(k\lambda z \int_{-\ell}^{+\ell} \frac{dx'}{[x'^2 + z^2]^{3/2}}\right)\hat{k}$$

recognize x'^2 = even

x' = odd

so

$$\int_{-\ell}^{+\ell} (\text{odd})\, dx = \cancel{} = 0$$

so $\vec{E}(z) = \left[k\lambda z \int_{-\ell}^{+\ell} \frac{dx'}{[x'^2+z^2]^{3/2}} \right] \hat{k}$ direction

ah... use $\int_{-\ell}^{+\ell} (\text{even})\,dx = 2\int_0^{\ell} (\text{even})\,dx$

$$= \left(2k\lambda z \int_0^{\ell} \frac{dx'}{[x'^2+z^2]^{3/2}} \right) \hat{k}$$

I'd try
int by substitution
$u = x'^2 + z^2$
.... oops won't work

will get to in a bit

GREAT steps in purple

try slowing down & exploiting symmetry

here is the problem again

Steps to Follow:

1.) coord syst (x,y,z)

2.) $dq = \lambda dx$

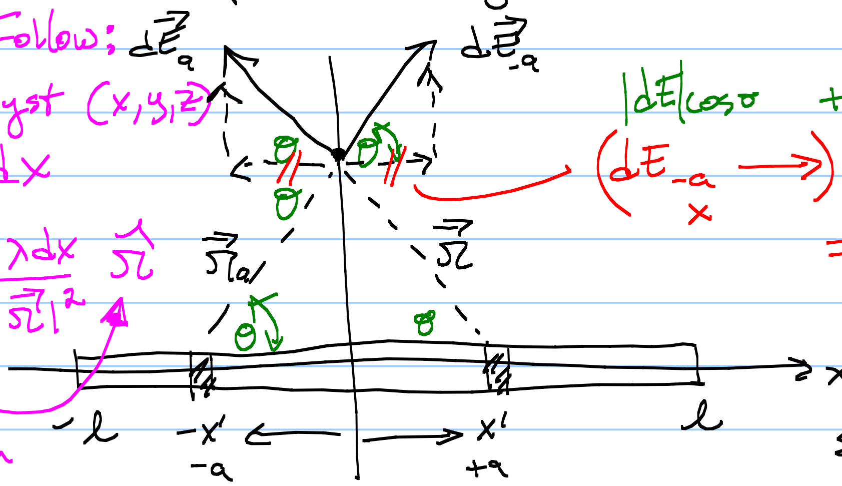
3.) $d\vec{E}(x) = \frac{K\lambda dx}{|\vec{\mathscr{r}}|^2}\hat{\mathscr{r}}$

4.) Now argue from symmetry that only component of $|dE|$ to survive is the z component so want

$|dE_z| = |dE|\sin\theta$

$= \left(\frac{K\lambda dx}{|\mathscr{r}|^2}\right)(\hat{z})\sin\theta$

5.) Now get $\mathscr{r}, \theta$ in terms of x, y, z



$|dE|\cos\theta \quad + -|dE|\cos\theta$

$\left(dE_{-a\,x} \longrightarrow\right) + \left(dE_{a\,x} \longleftarrow\right) = 0$

$\sin\theta = \frac{z}{|\mathscr{r}|} = \frac{z}{(x^2+z^2)^{1/2}}$

Notice, from symmetry ALL x-components cancel

$|\vec{\mathscr{r}}_a| = |\vec{\mathscr{r}}_{-a}| \quad = \vec{r} - \vec{r}' = (0,0,z) - (x,0,0)$

so

$|dE_a| = \left|K\frac{dq}{\mathscr{r}_a^{\,2}}\right| = \left|K\frac{dq}{\mathscr{r}_{-a}^{\,2}}\right| = |dE_{-a}|$

And z-components add $\Leftarrow$ So by symmetry, noticing can take $\hat{z}$ out of integral & in essence integrate only the magnitude

$= (2)|dE|\sin\theta\,\hat{z}$

so $\vec{E}(z) = \left[\int_0^{+l} 2K\frac{\lambda\,dx'}{[x'^2+z^2]^2}\left(\frac{z}{(x'^2+z^2)^{\frac{1}{2}}}\right)\right]\hat{z}$

or $\vec{E}(z) = \left[2\lambda K z \int_0^{+l} \frac{dx'}{\left[x'^2 + z^2\right]^{3/2}} \right] \hat{z}$ ⇐ Same as Before!

which way is easier? you must decide

Now to finish. Need to integrate

substitution
parts
dis and int

OK may Trig Sub? ⇐ ask Brian

For Now, Maple:

int(z/(x^2+z^2)^(3/2), x=0..l);

=

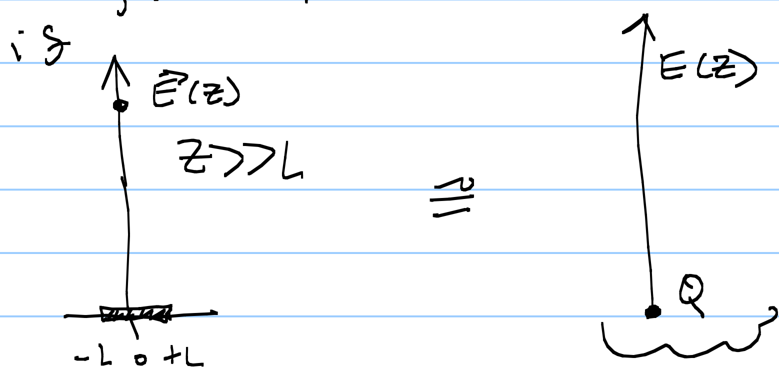
$$\vec{E}(z) = 2\lambda K z \left(\frac{L}{z^2 (L^2+z^2)^{1/2}} \right) (\hat{z})$$

$$\vec{E}(z) = K \frac{2\lambda L}{z\left(z^2+L^2\right)^{1/2}} \left(\hat{z}\right)$$

$\boxed{\text{DONE}}$ $\Rightarrow$ NO way ... have to ask yourself, is it RIGHT?

Well, what do we know?

is



where simple

$$\left|\vec{E}(z)\right| = K \frac{Q}{z^2}$$

NOTE: Considering $z >> L$ But not nec $z >> \infty$: $z = \infty$ $|E(z)| = 0$

so don't want that Limit Instead look @ a series expansion!

So, lets look @ $\lim\limits_{z >> L} E(z)$

$$= K\, 2\lambda L \left(\frac{1}{z} \frac{1}{z\left(1+\left(\frac{L}{z}\right)^2\right)^{\frac{1}{2}}} \right) \hat{z}$$

now $z >> L$, $(1+x)^m$; $x << 1$

$= 1 + mx + \dots\dots$

Binomial series expansion

$$= K\, 2\lambda L\, \frac{1}{z^2}\left(1+\left(\frac{L}{z}\right)^2\right)^{-\frac{1}{2}}\ \hat{z}$$

$$\simeq K\, 2\lambda L\, \frac{1}{z^2}\left(1 + -\frac{1}{2}\left(\frac{L}{z}\right)^2 + \text{H.O.T}\right)\hat{z}$$

$$\simeq K\, \frac{\lambda\, 2L}{z^2}\, \hat{z} \qquad ;\ \lambda\, 2L = Q_{TOT}$$

$$= K\, \frac{Q_T}{z^2}\, \hat{z}$$

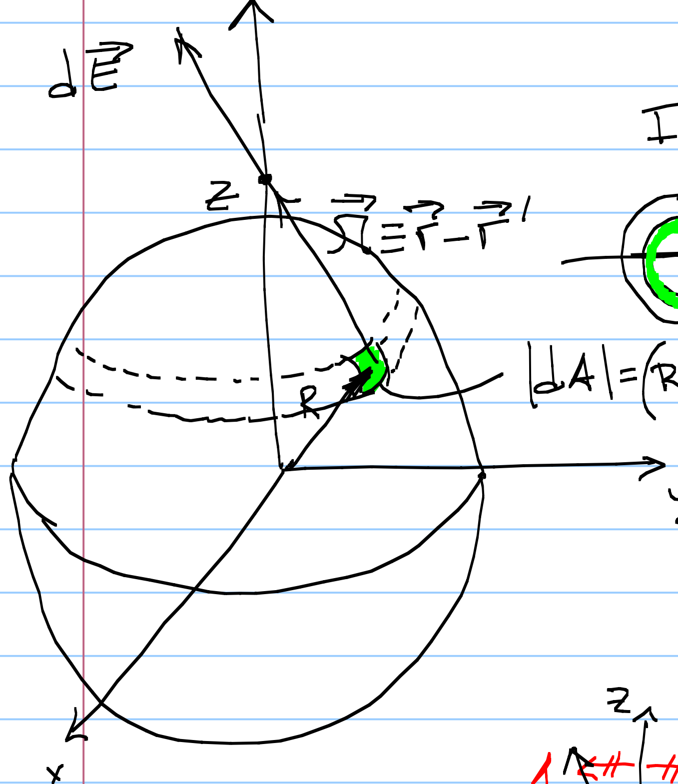
YES!

At least on the right track!

Another

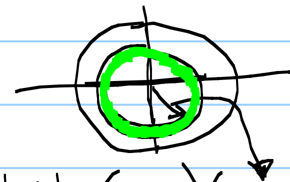
prob 2.7 Griffs

see next page For great steps



$|dA| = (R\,d\theta)(R\sin\theta\,d\phi)$

Find $\vec{E}(z)$

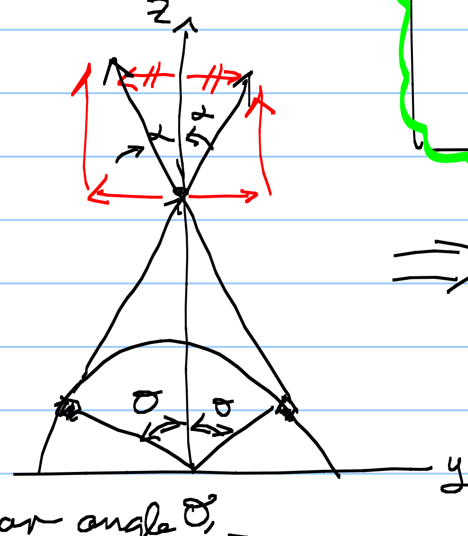


(z = dist from center of sphere, rad R w/ uniform surface charge density

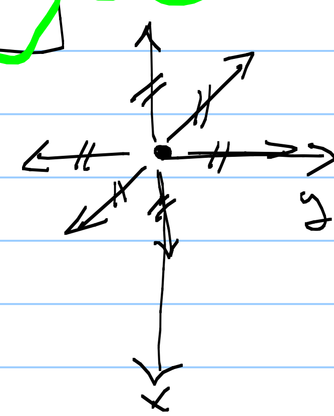
Always start w/

$$d\vec{E}(z) = K\frac{dq}{|\mathscr{r}|^2}\hat{\mathscr{r}} = \frac{K\,\sigma\,dA'}{|\mathscr{r}|^2}\hat{\mathscr{r}}$$

Can see that



$\Longrightarrow$ Top



@ a given polar angle θ, all components of $d\vec{E}(z)$ cancel in the x-y plane leaving only $d\vec{E}$ in $\hat{z}$ dir.

$$|dE_z| = |dE|\cos\alpha$$

Steps! Same as other problem

1.) choose coord syst (R, θ, ϕ) ; $R = \text{const}$

2.) dq : $dq = \sigma dA = R^2 \sin\theta \, d\theta \, d\phi$

3.) now
$$d\vec{E}(z) = \frac{K\sigma R^2 \sin\theta \, d\theta \, d\phi}{|\vec{\mathscr{r}}|^2} \hat{\mathscr{r}}$$

4.) Argue

That x & y components cancel exactly so left w/ just z-component $\hat{z}$

5.) Now Find $\left| d\vec{E}(z) \right|_z = \left| dE(z) \right| \cos\alpha$

6.)
$$d\vec{E}(z) = \left[\frac{K\sigma R^2 \sin\theta \, d\theta \, d\phi}{|\vec{\mathscr{r}}|^2} \cos\alpha \right] \hat{z}$$

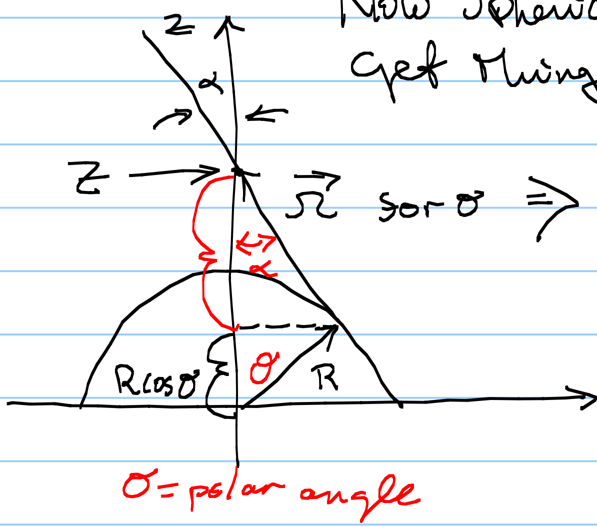
7.) get all variables in Terms of R, θ, ϕ

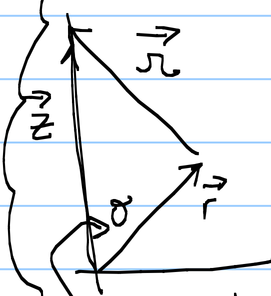
So need to sum up contributions for given θ

$$d\vec{E}(z) = \frac{K\sigma\, dA'}{|\mathscr{r}|^2}\hat{\mathscr{r}} \Rightarrow \frac{K\sigma\, dA'}{|\mathscr{r}|^2}\cos\alpha\, \hat{z}$$

Now Spherical Symmetry so try to get things in terms of (R, θ, ϕ)

$R = \text{const}$





$\vec{\mathscr{r}} = \vec{z} - \vec{r}$

$|\mathscr{r}| = \sqrt{\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}}$

$|\mathscr{r}|^2 = \left(z^2 + r^2 - 2\vec{z}\cdot\vec{r}\right)$

$|\mathscr{r}|^2 = z^2 + R^2 - 2zR\cos\theta$

cause $r^2 = \text{mag} = R^2$

so $$\cos\alpha = \frac{z - R\cos\theta}{|\mathscr{r}|}$$

$$= \frac{z - R\cos\theta}{\left[z^2 + R^2 - 2zR\cos\theta\right]^{1/2}}$$

$$\underset{\text{@ fixed } \theta}{dE(z)} = \frac{K\sigma (R\,d\theta)(R\sin\theta\, d\phi)}{\left[z^2 + R^2 - 2zR\cos\theta\right]}\left(\frac{z - R\cos\theta}{\left[z^2 + R^2 - 2zR\cos\theta\right]^{1/2}}\right)\hat{z}$$

$$= \frac{K\sigma R^2 \sin\theta \left(z - R\cos\theta\right) d\theta\, d\phi}{\left[z^2 + R^2 - 2zR\cos\theta\right]^{3/2}}\,\hat{z}$$

$$E(z) = \left\{ \underbrace{\int_0^{2\pi}}_{\text{azimuthal}} \left[\underbrace{\int_0^{\pi}}_{\uparrow\,\text{polar}} \left(\quad \right) \right] d\phi \right\} \hat{z}$$

$$= \left(2\pi K\sigma R^2 \int_0^{\pi} \frac{\sin\theta \left(z - R\cos\theta\right) d\theta}{\left[z^2 + R^2 - 2zR\cos\theta\right]^{3/2}} \right) \hat{z}$$

Substitution?

$u = \cos\theta$ → $\pi,\ -1$; $0,\ 1$

$du = -\sin\theta\, d\theta$

old   new

$$= \left(\quad \right) \int_1^{-1} \frac{(z - Ru)\, du}{\left[z^2 + R^2 - 2zRu\right]^{3/2}}\,\hat{z}$$

Better

$$= \left(\quad \right) \left[\int_1^{-1} \frac{z\, du}{\left[z^2 + R^2 - 2zRu\right]^{3/2}} - \int_1^{-1} \frac{Ru\, du}{\left[z^2 + R^2 - 2zRu\right]^{3/2}} \right] \hat{z}$$

$$= \left(2\pi K \sigma R^2\right)\left[z \int_1^{-1} \frac{du}{\left[z^2+R^2-2zRu\right]^{3/2}} - R \int_1^{-1} \frac{u\,du}{\left[z^2+R^2-2zRu\right]^{3/2}} \right] \hat{z}$$