

E.F. Deveney / BSC Physics PH438 Tensors

Note Title

9/13/2004

Vectors: mag + dir

$$\vec{A} = A_x \hat{i} + A_y \hat{j} = (A_x, A_y) \text{ in matrix rep.}$$

now

$$a\vec{A} = a \left(\begin{array}{c} A_x \\ A_y \end{array} \right) = \begin{array}{c} \vec{aA} \\ aA_y \\ aA_x \end{array} = aA_x \hat{i} + aA_y \hat{j}$$

$$\text{mag} \Rightarrow a(\sqrt{A_x^2 + A_y^2}) = a|A| \quad \text{mag} \sqrt{a^2 A_x^2 + a^2 A_y^2} = a \sqrt{A_x^2 + A_y^2} \\ = a|A|$$

$$\text{dir} = \tan^{-1} \frac{A_y}{A_x}$$

$$\text{dir} = \tan^{-1} \frac{aA_y}{aA_x} = \tan^{-1} \frac{A_y}{A_x}$$

so

$\vec{B} = a\vec{A}$ = Same dir, different magnitude

in matrix form

$$a = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

$$\text{so } a\vec{A} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} aA_x + 0 \\ 0 + aA_y \end{pmatrix} = \begin{pmatrix} aA_x \\ aA_y \end{pmatrix} \\ = (aA_x, aA_y)$$

diff mag Same thing BUT same dir

Matrices

So cool, ^ puts vectors & scalars on same footing

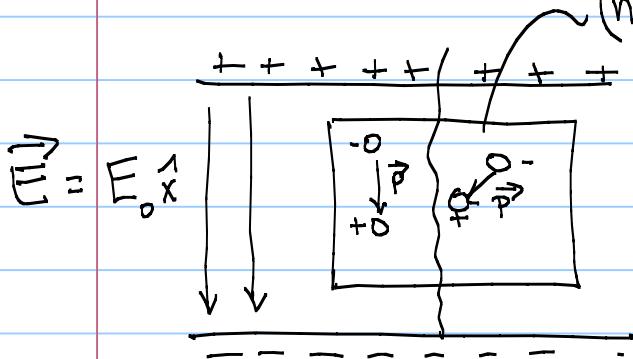
Now indeed vector cross product $\vec{C} = \vec{A} \times \vec{B}$ yields a new direction, \perp to $\vec{A} \& \vec{B}$.

But is there an object that "operates" on a vector, say \vec{A} , that changes the direction of \vec{A} ?

Well $\vec{B} = a\vec{A}$, \vec{B} is same dir as \vec{A}

want $\vec{B} = \underbrace{\text{?}}_{\vec{Q}} \vec{A}$, such that $\vec{B} \neq \text{dir of } A$

EXAMPLE!



(non linear polarizability)

linear: apply E_0 & $P \downarrow$

non linear: apply E_0 & $P \swarrow$

due to 'lattice' Force's the alignment is not complete!

so $\vec{P} = \underbrace{\alpha}_{\text{?}} \vec{E}$ such that $\vec{P} \text{ not } \parallel \vec{E}$

$$\text{try } \underline{\alpha} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} \\ \alpha_{yx} & \alpha_{yy} \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & 0 \\ \alpha_{yx} & 0 \end{pmatrix}$$

Then $\vec{E} = E_0 \hat{x} + 0 \hat{y}$

$$\vec{p} = \begin{pmatrix} \alpha_{xx} & 0 \\ \alpha_{yx} & 0 \end{pmatrix} \begin{pmatrix} E_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_{xx} E_0 \\ \alpha_{yx} E_0 \end{pmatrix}$$

or

$$\vec{p} = \alpha_{xx} E_0 \hat{x} + \alpha_{yx} E_0 \hat{y}$$

Clearly $\vec{p} \neq \parallel \text{ to } \vec{E}$

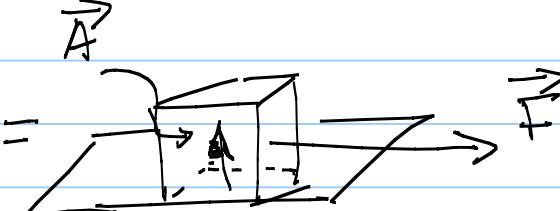
$\underline{\alpha} \Rightarrow$ "mixed" $E \hat{x}$ into $E \hat{y}$

Another example (s)

$\underline{\underline{S}}$ = Stress Tensor = $\frac{\vec{F}}{A}$ = pressure
 not well defined cause both \vec{F} & A have directions!

so $\underline{\underline{S}}$ = generalization of pressure

\rightarrow 1) compression/
 tension $\equiv \frac{F_{\perp}}{A}$
 \rightarrow 2) shear $\equiv \frac{F_{||}}{A}$

Shear $=$  $= \frac{F_{\rightarrow}}{A_{\text{app}}}$

'resist pull'
 'Layers appart'

pressure (comp Tension) $=$  $= \frac{F_{\uparrow}}{A_{\uparrow}}$

$$\underline{\underline{S}} = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix} = \begin{pmatrix} \frac{F_x}{A_x} & \frac{F_x}{A_y} \\ \frac{F_y}{A_x} & \frac{F_y}{A_y} \end{pmatrix}$$

or:

— 1-D space \hat{X}

(m) , $m = \text{mass}$



$$\hat{X}' = \underbrace{m}_{\uparrow} \hat{X}$$

\underbrace{m} warps \hat{X} into 2-D space

or generalize to 3-D

so

\underbrace{m} = metric tensor (?)

Grecot... Scalars, vector, Tensors \rightarrow Matrix
~~~~~  
or  
'mixing'  
matrix  
Rep  
handy.

Need to be more formal!  
Very Sophisticated!

Idea: True physical objects or Laws are  
defined in "How they transform"

Because

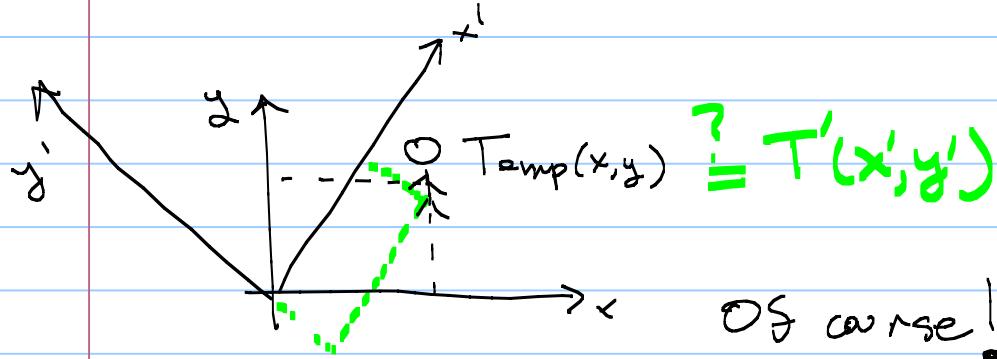
as we know, Big assumption in  
physics is there is no perfect  
place,  $x_0$   
angle,  $\theta_0$   
time,  $t_0$

So to study objects & laws

- 1.) True laws should have some "form" mathematical  
in all transformations (invariant)  
most form  
shouldn't change  
go to  $\pi$
- 2.) Objects of the same class (vectors) should  
all change in the same way if in  
fact they are the same thing as  
each other! IS it transforms like

a scalar (or vector) it is!

So: How do Scalars Transform?



Even is

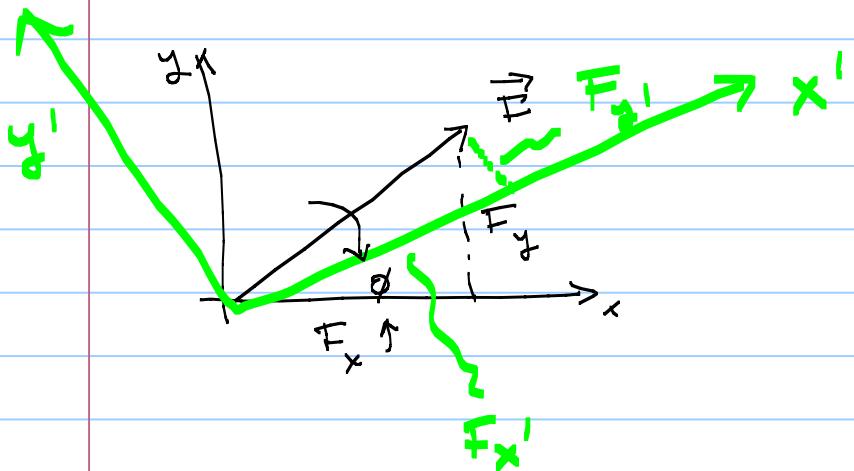
$$N = (N_{\text{apples}} + N_{\text{bananas}}) \stackrel{?}{=} (N'_a + N'_b) \quad x, y \quad x', y'$$

of course! So even <sup>(in)</sup> "components"  
scalars are invariant to  
coordinate transformations

is  $M(x, y, z) = M'(x', y', z')$   $\Rightarrow$  its a scalar!

a scalar is called Tensor Rank 0

OK, How do vectors transform?



Note: TRANSFORMATION HERE is called orthogonal,  $\perp$ , you don't have to do  $\perp$  transformation but extremely useful

1) goes from one  $\perp$  set  $(x, y)$  to new  $\perp$  set  $(x', y')$

2) PRESERVE vector magnitude!

3.) maintains  $\overset{\text{scalar}}{\text{dot - product}}$

In physics, if idea is right, free to choose coord system to choose  $\perp$  coord systems related by orthogonal transformations /

$SO(2)$  = Special orthogonal

group = group of

2-D transformations or rotations that preserve 2D

Group Theory  $\implies$

extremely useful idea in Q.M.  
as

State ket  $|\Psi\rangle$  are normalized  
in the

Schrö-Born probabilistic interp

$$\psi^* \psi = \text{prob density.}$$

Since  $|\Psi\rangle = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$  can be thought  
of as vector w/  $c_+, c_- \in \mathbb{C}$  complex  
(thus CAVS)

FREE to do Quantum mechanics in most  
convenient CAVS related by

The equiv of orthog Transformations

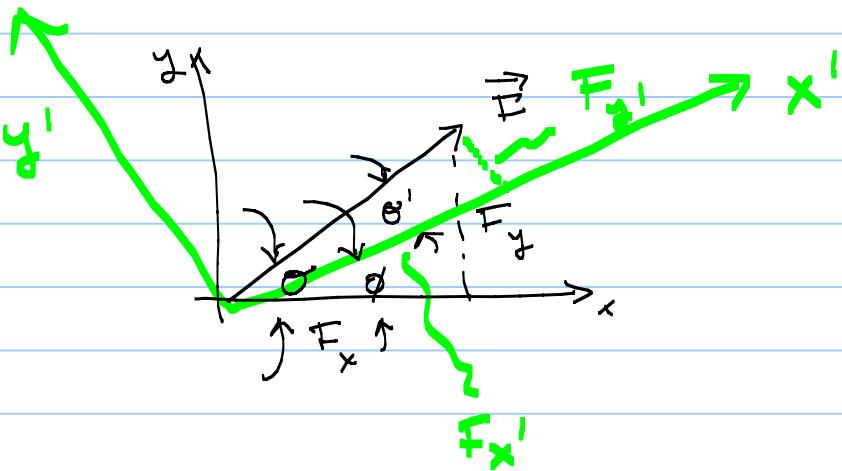
for  
Complex vectors called UNITARY  
transformations

↙ The group is called  $SU(2n^3)$  preserving

Special Unitary transformation group ---  $\int \psi^* \psi = 1$   
probability

Back to ....

OK, How do vectors transform?



can show ...

$$F_x = F \cos \phi$$

$$F_y = F \sin \phi$$

$$\begin{aligned} F_{x'} &= F \cos \phi' = F \cos(\phi - \phi) = F [\cos \phi \cos \phi + \sin \phi \sin \phi] \\ &= F_x \cos \phi + F_y \sin \phi \end{aligned}$$

$$\begin{aligned} F_{y'} &= F \sin \phi' = F \sin(\phi - \phi) = F (\sin \phi \cos \phi - \cos \phi \sin \phi) \\ &= F_y \cos \phi - F_x \sin \phi \end{aligned}$$

or

$$\begin{pmatrix} F_{x'} \\ F_{y'} \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} F_x \cos \phi + F_y \sin \phi \\ -F_x \sin \phi + F_y \cos \phi \end{pmatrix}$$

or  $\vec{F}' = R \vec{F}$

$$\text{now } \vec{t}' = \underbrace{\vec{R}}_{\uparrow} \vec{t}$$

$\vec{R}$  doesn't have to, but here is an orthogonal transformation: which here convenient to call

orthogonal rotation matrix  
which has all the special properties (preserves dot prod & vector mag)

$$+ \quad \vec{R}^T = \vec{R}^{-1}$$

So

if you have matrix  $\vec{R}^T = \vec{R}^{-1}$  you've got a  $\perp$  rotation matrix!

In Q.M. recall you have CAVS so requirements

$$\vec{R}^{T*} = \vec{R}^{-1} ; \text{ say } \vec{R} = \text{unitary}$$

$$\vec{R}^+ = \vec{R}^{-1}$$

Backing up

$$\vec{F}' = R \vec{F}$$

$R$  defines the transformation

$$\begin{pmatrix} F_x' \\ F_y' \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} R_{xx}F_x + R_{xy}F_y \\ R_{yx}F_x + R_{yy}F_y \end{pmatrix}$$

or That

$$F_i' = \sum_{j=1}^n R_{ij} F_j \quad * \text{ note abstract}$$

$x \rightarrow 1$

$y \rightarrow 2$

$z \rightarrow 3$

so

$$F_x' = F_i' = \sum_{i=1}^n R_{xi} F_x + R_{yi} F_y$$

Sum over  $j$

YUP!

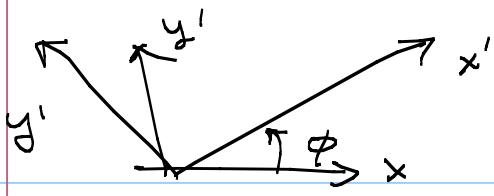
So: If  $M(x,y) = M'(x',y')$  = scalar

If Components of an object transform like

Einstein summations for  
convention indices  
repeated vector  $=$   
 $\Rightarrow i \sum_i$

$$F_i' = \sum_j R_{ij} F_j \iff \text{Then } F_i \text{ are components of a VECTOR.}$$

Tensor Rank 1



$\underline{R}$  defines transformation

Scalar

$$M'(x', y') = (\underline{R})^{\circ} M(x, y)$$

=  $M(x, y)$  = Tensor, Rank 0

How many times  
had to use  $\underline{R}_1$   
which defines rotation

Vector

$$\vec{F}(x', y') = \vec{F}(x, y)$$

But need components to  
change as

$$\vec{F}'_i = \sum_j R_{ij} F_i$$

Tensor  
RANK

1

need to use  
 $\underline{R}_1$  once!

$$\text{Now } S_i = \begin{pmatrix} \frac{F_x}{A_x} & \frac{F_y}{A_y} \\ \frac{F_y}{A_x} & \frac{F_y}{A_y} \end{pmatrix}$$

can imagine that each  
component,  $S_{xx} = \frac{F_x}{A_x}$  requires  
2  $\underline{R}_1$ 's to transform

$$\text{Indeed } S_{ij}' = \sum_k \sum_l R_{ik} R_{jl} S_{kl} = \text{Tensor Rank 2}$$

$\underline{R}_{ik} \underline{R}_{jl} S_{kl}, \sum_{kl} \text{ implied}$

Note that

$$\vec{A} = (A_x, A_y) = \underbrace{\Pi R_1}_{\text{matrix}}, \text{ 2-D } 1 \times 2$$

$$\text{while } \vec{A} = (A_x, A_y, A_z) = \underbrace{\Pi R_1}_{\text{matrix}}, \text{ 3-D } 1 \times 3$$

doesn't  
change!

$$\xi \quad \vec{S} = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix} = \underbrace{\Pi R_2}_{\text{matrix}}, \text{ 2-D } 2 \times 2$$

$$\text{or} \quad \vec{S} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix} = \underbrace{\Pi R_2}_{\text{matrix}}, \text{ 3-D } 3 \times 3$$

RANK really about how many times you need

$R_i$  for each component ie ( $S_{xy} = \frac{F_x}{A_x}$ )

Size of matrix = # of Components ex: 2D,  $\vec{F} = F_x, F_y$

Can you build higher order

Tensors  $\mathbb{R}^3$

:  
y up!

Matrix rep let you treat all  
objects as some shooting

$$a = a \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbb{R}^D$$

$$F = (F_x, F_y), \mathbb{R}^I$$

$$S = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix}, \mathbb{R}^2$$

Using matrix Algebra!

THUS  $\Rightarrow$  good Math software

package are Based on Matrix

Algebra  $\xrightarrow{\text{maple}}$   
 $\xrightarrow{\text{mathematica}}$