

E.F. Devaney / BSC Physics Vector Integration Gauss Stokes

Note Title

9/25/2004

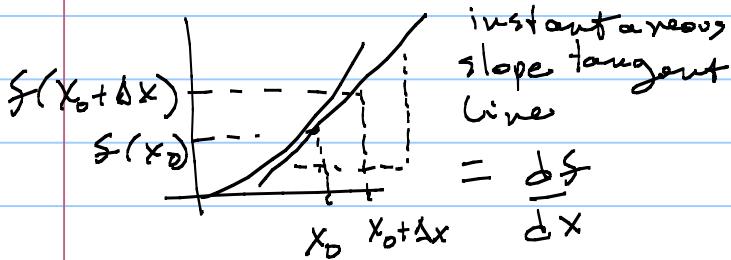


Fund Theorem

Gauss

Stokes

So we have 1-D diff & Int calc



such that

$$S(x_0 + \Delta x) \approx S(x_0) + \frac{dS}{dx} \Big|_{x_0} \Delta x$$

or

$$dS = S(x_0 + \Delta x) - S(x_0) = \left(\frac{dS}{dx} \right) dx$$

in limit

$$\Delta x \rightarrow dx$$

finite infinitesimal

so

$dS = \text{total}$

$dS = \left(\frac{dS}{dx} \right) dx \equiv \text{infinitesimal change in } S(x)$

even more
from x to $x + dx$

2) The Integral

\therefore If you add up

The infinitesimal changes from

$$S(a) \text{ to } S(b)$$

This is
Fundamental
Theorem of Calc!

you get The TOTAL change!

$$\Delta S = \sum_a^b dS$$

$$= S(b) - S(a)$$

Total Change is S from a to b is

$$S(b) - S(a) = \sum_a^b \left(\frac{dS}{dx} \right) dx$$

where $S \Rightarrow \sum_{\substack{\text{over} \\ \text{infinitesimal} \\ \text{changes}}} \left(\frac{dS}{dx} \right) dx$

This my famous: $\text{Sd}[\text{elephant}] = \text{elephant!}$

Now to Vector Calc. (V.C.)

Recall 3 derivs in V.C.

Based on $\vec{\nabla} \equiv \vec{\Delta L}$

$\text{Grad} = \vec{\nabla} \phi$

$\text{Div} = \vec{\nabla} \cdot \vec{v}$

$\text{Curl} = \vec{\nabla} \times \vec{v}$

So we expect 3 Fund Theorems, related to
3 types of integrals!

In general can think of 3 types
But all same idea

$\int_{\text{Space}} [\text{deriv of a vector}] d(\text{region}) = \text{Sum of the vector evaluated at the perimeter of the region!}$

$\int_{\text{path}} = \text{eval @ end pts} \cdot \vec{a}, \vec{b} = \text{end pts}$

$\int_{\text{Area}} = \text{eval over line (perimeter of A)}$

$\int_A = \text{eval over Area enclosing A}$

$\int_S = \text{eval over Surface Area}$

I) Fund Theorem of Gradients $\int_{\text{line}} S = \text{end pts}$

$$\text{recall } dT = (\vec{\nabla} T) \cdot d\vec{l} = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

sum of partial changes

$\therefore dT = \text{[infinitesimal total change] in } T(x, y, z) \text{ moving}$

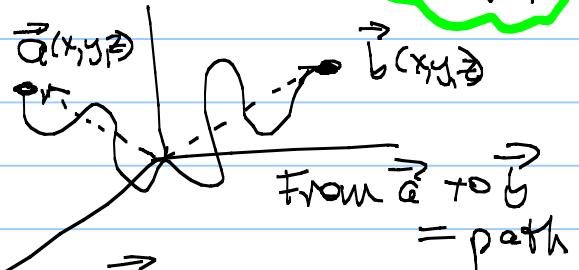
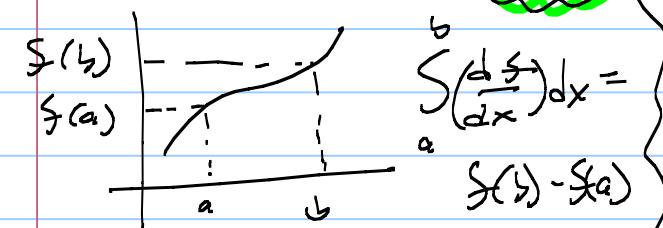
$$\text{now } d\vec{l} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

Just as in 1-D Fund Theorem of Calc,

The total change in dT $\xrightarrow{\text{from}} \text{vector position } \vec{a} \xrightarrow{\text{to}} \text{vector position } \vec{b}$

is sum of infinitesimal changes!

look 1-D calc $\int_a^b \frac{dS}{dx} dx = S(b) - S(a)$



$$\text{So } T(\vec{b}) - T(\vec{a}) = \int_{\substack{\vec{a} \\ (\text{path})}}^{\vec{b}} dT = \int_{\substack{\vec{a} \\ (\text{path})}}^{\vec{b}} (\vec{\nabla} T) \cdot d\vec{l}$$

so $\int_{\vec{a}}^{\vec{b}} (\nabla T) \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$
 path

= Find Theorem of gradients

What does it look like?



See next page

$$\begin{aligned} \vec{b} &= x_s \hat{x} + y_s \hat{y} + z_s \hat{z} \\ \vec{a} &= x_i \hat{x} + y_i \hat{y} + z_i \hat{z} \\ &= \int_{x_i, y_i, z_i}^{x_s, y_s, z_s} \left(\frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right) \cdot (dx \hat{x} + dy \hat{y} + dz \hat{z}) \end{aligned}$$

CAREFUL! Not @ all clear that can just do

$$\int_{x_i}^{x_s} \frac{\partial T}{\partial x} dx + \int_{y_i}^{y_s} \frac{\partial T}{\partial y} dy + \int_{z_i}^{z_s} \frac{\partial T}{\partial z} dz$$

because what is $T = x^2 y^2$, then $\frac{\partial T}{\partial x} = 2xy^2$ what do you do w/ y?

Trick

$$\int_{\text{path}} = \text{evaluated } @ \text{ point vector elements}$$

That path is not unique! Just hit

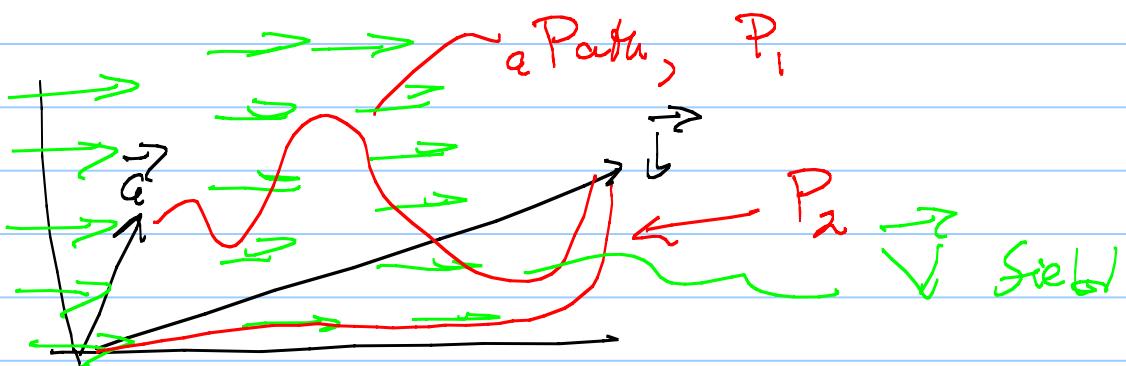
Great!

$$\oint_{\text{closed loop}} \vec{V} \cdot d\vec{l} : \vec{V} = \text{vector field}$$

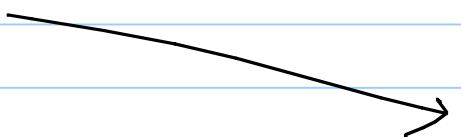


The integral from \vec{a} to \vec{b} doesn't mean anything until you

"Specify what path you want to take thru the field"



Now... Pg 26 shows that

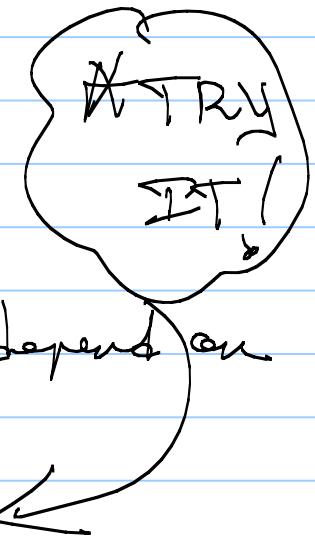
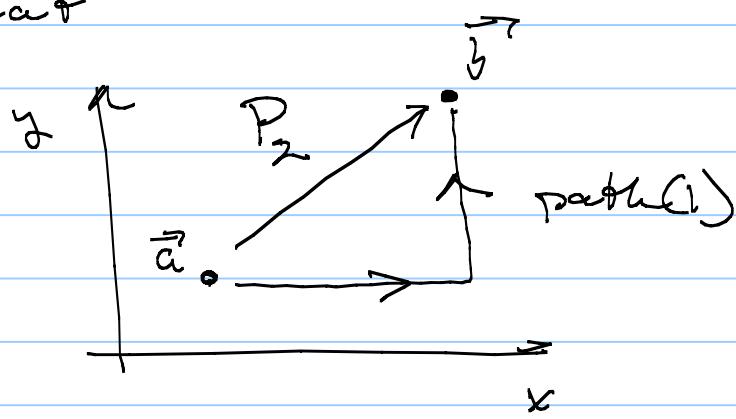


Pg 26, ex: 1.6

$$\text{If } \vec{V} = y^2 \vec{i} + 2x(y+1) \vec{j}$$

$$\text{and } \vec{a} = (1, 1, 0), \vec{b} = (2, 2, 0)$$

That



$\int_{\vec{a}}^{\vec{b}} \vec{V} \cdot d\vec{b}$ does indeed depend on
the path!

It is generally TRUE,

However

If $\vec{V} = \nabla T$; ie the vector field

\vec{V} is from the grad of a scalar

if $\vec{V} = \vec{\nabla}T$,

Then

$$\int_{\vec{a}}^{\vec{b}} (\vec{\nabla}T) \cdot d\vec{l} \quad \text{IS NOT Path Dependent!}$$

$\therefore \int_{\vec{a}}^{\vec{b}} \vec{V} \cdot d\vec{l}$ in general = path dep

but

$\int_{\vec{a}}^{\vec{b}} \vec{V}_c \cdot d\vec{l}$ is NOT path depend

where $\vec{V}_c = \vec{V}_{\text{conservative}} = \vec{\nabla}T$

So Indeed

$$\int_{\vec{a}}^{\vec{b}} (\vec{\nabla}T) \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$$

Indep of Path!

Now you could choose a path that does indeed hold $y = \text{constant}$ while x varies

such that

$$\int_{x_i}^{x_s} \left(\frac{\partial T}{\partial x} \right) dx$$

$$\int_{x_i}^{x_s} (2xy^2) dx$$

$y = \text{constant}$ | while x varies

so can do "the integral!"

OR! is
could find
path where
know how
 y changes w/ x
ie get $y(x)$

\int_{∂} :

$$T(\vec{s}) - T(\vec{c}) = \sum_{x_i, y_i, z_i} \left(\frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz \right)$$

2 KEY Corollaries!

i) $\int_{\vec{a}}^{\vec{b}} (\vec{\nabla} T) \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$ is indep of path from \vec{a} to \vec{b}

since $\vec{\nabla} T = \text{Grad } T = \text{vector can rewrite as}$

$$\int_{\vec{a}}^{\vec{b}} \vec{V} \cdot d\vec{l}$$

It's you FIND $\int_{\vec{a}}^{\vec{b}} \vec{V} \cdot d\vec{l}$ = indep of path
from $\vec{a} \rightarrow \vec{b}$ Then

clearly
 \vec{V} can be expressed as $\vec{\nabla} T$

Huge! such vectors, that follow this &
can be expressed as Gradients of
scalars are called Conservative.

turns out all forces in nature have this property & one :-

1) conservative

2) get from ∇T where scalar T is called potential

ex: gravitational potential
electrical potential

* Foresight to E&M!

tough to solve so

\vec{E} = vector Electric field cause it's a vector

$$\propto \int \left(\frac{1}{r^2}\right) dr \hat{r}$$

while ϕ = electric potential

$$\text{where } \vec{E} = -\nabla \phi$$

$$\oint \phi d\vec{r} \propto \int \frac{1}{r} dr$$

no vector
so easier by Far!

$$\begin{aligned}\vec{x}_s &= \vec{x}_i \\ T(\vec{x}_s) - T(\vec{x}_i)\end{aligned}$$

Corollary 2: $\oint (\vec{\nabla} T) \cdot d\vec{l}$ has $\tau_0 = 0$

when $\oint \rightarrow \sum_{\vec{a}}^{\vec{a}}$ thru any path

Back to potentials

$$So : \oint (\vec{\nabla} T) \cdot d\vec{l} = 0$$

or

$$\oint \vec{F} \cdot d\vec{l} = 0$$

Another idea of, or clue, that have
Conservative Forces $\Leftrightarrow \vec{F}_c = \vec{\nabla} V$
 $= \text{grad of some scalar potential}$

Then

$$\text{recognize } \oint \vec{F} \cdot d\vec{l} = \text{work}$$

Therefore is 1) work is indep of path
2) work around any closed loop = 0

Then Force = again Conservative & can
find some potential such that $\vec{F}_c = \vec{\nabla} V$
& it's way easier to work with

note: all forces = conservative \Rightarrow

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0$$

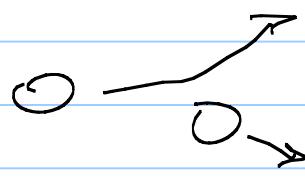
So means no friction!

Seems contradictory BUT @ lowest level:

1) For Big Things

Friction & entropy $\&$ inability to track
the total work of 10^{23} particles

2) For little things

 (ie)
easy to keep track of
everything $\&$

'Losses' due to
friction are gone!

Griss $\int (\nabla T) \cdot d\vec{l}$ Fund Theory of
 Gradients! $\nabla T =$
 (max & dir)
 Space rate
 of change
 of T

I.

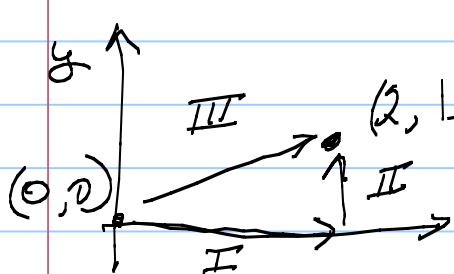
$$\text{Ex: 1.9 } T = xy^2; \vec{a} = \text{origin } (0,0,0) \\ \vec{b} = (2,1,0)$$

$$\int_{(0,0,0)}^{(2,1,0)} (\nabla T) \cdot d\vec{l} = \int_{(0,0,0)}^{(2,1,0)} (y^2 dx + 2xy dy)$$

what y ? T T
 what to do? what x where y varies?

Concentrate on the path!

notice, really from $(0,0)$ to $(2,1)$ cause
 $z = \text{constant}$ so 2 ideas (Griss)



try $P_1 = I + II \Rightarrow \int_0^2 dx \text{ where } y=0$ const
 $+ \int_0^1 dy \text{ where } x=2$ const

slope III = $\frac{\Delta y}{\Delta x} = \frac{1}{2}$
 $\Rightarrow y = \frac{1}{2}x + 0$

2. $P_2 = III$ where $\int_0^2 dx \text{ of } y = \frac{1}{2}x$

1) $P_1 = I + II$

$$x_3 = 2$$

$$= \int_{x_i=0}^{x_3=2} (y=0) dx + \int_{y_i=0}^{y_s=1} 2(x=2)y dy$$

$$= 0 + 4 \left[\frac{1}{2} y^2 \right]_0^1$$

$$+ 2 = \boxed{2}$$

2) $P_2 = III$

$$\frac{2,1,0}{2,1,0}$$

$$\int_{0,0,0}^{2,1,0} (\vec{\nabla} T) \cdot d\vec{l} = \int_{0,0,0}^{2,1,0} (y^2 dx + 2xy dy)$$

↑ what is y?
↑ what x when y varies?

Try path 2 where

$$y = \frac{1}{2}x \quad \Rightarrow \quad dy = \frac{1}{2}dx$$

good!

$$\int \left[\left(\frac{1}{2}x \right)^2 dx + 2x \left(\frac{1}{2}x \right) \frac{1}{2} dx \right]$$

$$\int \left(\frac{1}{4}x^2 + \frac{x^2}{2} \right) dx = \int_0^2 \frac{3}{4}x^2 dx$$

$$= \frac{3}{4} \left[\frac{1}{3}x^3 \right]_0^2 = \frac{x^3}{4} \Big|_0^2 = \frac{8}{4} - 0 = \boxed{2}$$

Grads & Lw's, 2nd theorem of Grads

$$\int_{\vec{a}}^{\vec{b}} (\nabla T) \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$$

=

problem 6.31

II

2nd Vector deriv ($\vec{\nabla} \cdot \vec{V}$) Gauss theorem

Find Theorem of (Set) Divergence!

$\vec{V} \cdot \vec{V}$ & Divergence
dot or 'sourcing'
Sum it to pts

Interesting mathematically & will lead to new, refined, definition of $\vec{\nabla} \cdot \vec{V}$

1) $\int_{\text{path}} \vec{\nabla} T \cdot d\vec{l} = \text{evaluate at } \begin{matrix} \text{(vector)} \\ \text{end pts} \end{matrix}$

2) $\int_{\text{Area}} (\vec{\nabla} \times \vec{V}) \cdot d\vec{A} = \text{evaluate over path Bounding area}$

3) $\int_{\mathcal{V}} (\vec{\nabla} \cdot \vec{V}) dV = \text{evaluate over area Bounding } \mathcal{V}$

Follow Grifft, do next!

$$\int_{\mathcal{V}} (\vec{\nabla} \cdot \vec{V}) dV = \int_{\text{Surface Area enclosing } \mathcal{V}} \vec{V} \cdot d\vec{a}$$

called 1) Gauss's
2) Green's
3) Divergence Theorem!

using Gauss's
in Calc III
Calculus
* Saved my Butt!

But $\int \vec{V} \cdot d\vec{a}$
= easy if
see right
away
 $\vec{\nabla} \cdot \vec{V} = 0$
my calcs

Let's see if we can understand what a $\vec{\nabla} \cdot \vec{V}$ is more than what we had before

Proof $\int_F (\vec{\nabla} \cdot \vec{V}) d\tau = \int_S \vec{V} \cdot d\vec{a}$ by Example

1.0 in

Gauss

maybe
different

Ex: $\vec{V} = y^2 \hat{x} + (2xy + z^2) \hat{y} + (2yz) \hat{z}$

Here:

$$\int_F (\vec{\nabla} \cdot \vec{V}) d\tau$$

over unit cube

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial(y^2)}{\partial x} + \frac{\partial(2xy + z^2)}{\partial y} + \frac{\partial(2yz)}{\partial z}$$

$$= 0 + 2x + 2y$$

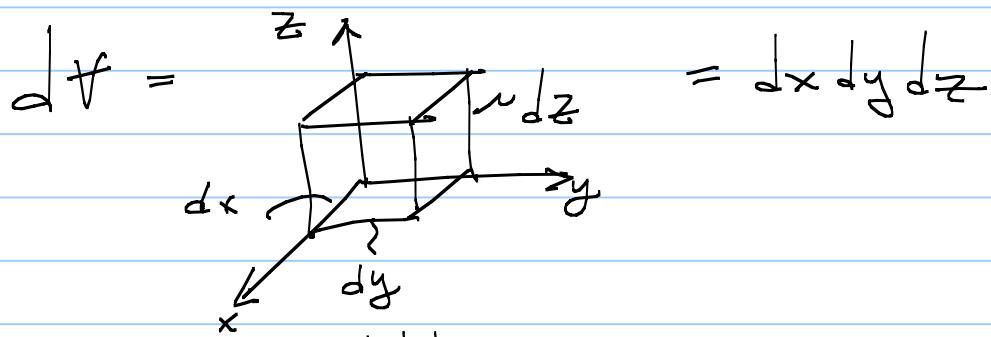
$$\vec{\nabla} \cdot \vec{V} = 2(x+y)$$

Now $d\tau = ?$

what coordinates

- 1) spherical
- 2) cylindrical
- 3) cartesian

cartesian good but clearly



$$\int_F (\vec{\nabla} \cdot \vec{V}) d\tau = \int_{0,0,0}^{1,1,1} 2(x+y) dx dy dz$$

$$= \int_{0,0}^{1,1} 2(x+y) dx dy$$

= 1

$$= \int_0^1 \left[\int_0^1 (x+y) dy \right] dx$$

$$= 2 \int_0^1 \left[x(1) + \frac{1}{2} y^2 \Big|_0^1 \right] dx$$

$$= 2 \int_0^1 (x + \frac{1}{2}) dx$$

$$= 2 \left[\frac{x^2}{2} \Big|_0^1 + \frac{1}{2} x \Big|_0^1 \right]$$

$$= 2 \left[\frac{1}{2} + \frac{1}{2} \right] = 2$$

in

$$\int_S \vec{\nabla} \cdot \vec{V} d\tau = 2$$

$$\hookrightarrow ? \int_S \vec{V} \cdot d\vec{\alpha}$$

This is
much harder!

* note quite different than line integral

$$\int_{0,0,0}^{2,1,0} (y^2 dx + 2xy dy)$$

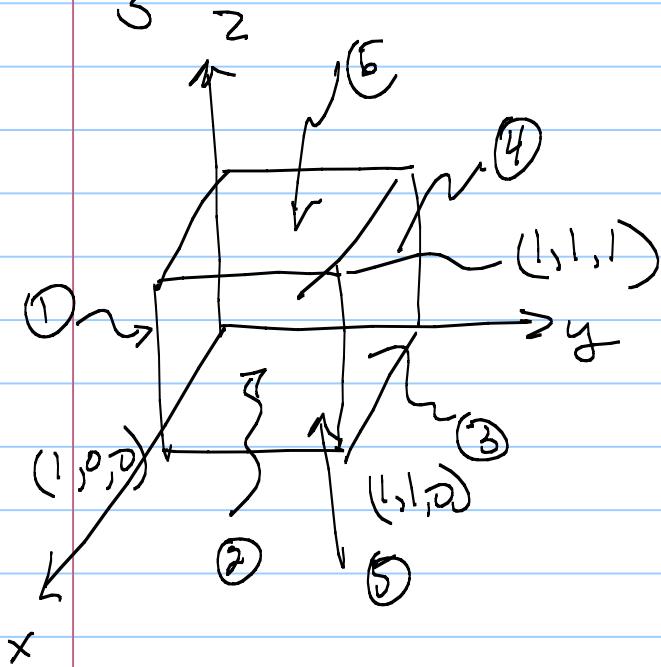
$$\int_{0,0,0}^{2,1,0} y^2 dx \sim \text{OK}$$

can integrate dx
but what y do you use?

here
you are integrating
over $dy \frac{d}{dx} dx$
which means
y has value $\frac{dy}{dx}$
 $\frac{d}{dx}$ x " varying dy

$$\vec{V} = y^2 \hat{x} + (2xy + z^2) \hat{y} + (2yz) \hat{z}$$

$\oint \vec{V} \cdot d\vec{a}$ means need $d\vec{a}$! ie 6 faces!



$$\oint_A \vec{V} \cdot d\vec{a} =$$

$$\begin{aligned} \oint_{A_1} \vec{V} \cdot d\vec{a}_1 &= \int_0^1 \int_0^1 (2xy + z^2)(1) dx dz \\ &= \int_0^1 \int_0^1 z^2 dx dz = -\frac{1}{3} \end{aligned}$$

$$\oint_{A_2} \vec{V} \cdot d\vec{a}_2 = \frac{1}{3}$$

$$\text{ex: } A_1$$

$$d\vec{a}_1 = dx dz (-\hat{y})$$

$$A_2: d\vec{a}_2 = dy dz (+\hat{x})$$

$$A_3: d\vec{a}_3 = dx dz (+\hat{y})$$

$$A_4: d\vec{a}_4 = dy dz (-\hat{x})$$

$$A_5: d\vec{a}_5 = dx dy (-\hat{z})$$

$$A_6: d\vec{a}_6 = dx dy (+\hat{z})$$

$$\oint_{A_3} \vec{V} \cdot d\vec{a}_3 = \frac{4}{3}$$

$$A_3$$

$$\oint_{A_4} \vec{V} \cdot d\vec{a}_4 = -\frac{1}{3}$$

$$A_4$$

$$\oint_{A_5} \vec{V} \cdot d\vec{a}_5 = 0$$

$$+$$

$$\oint_{A_6} \vec{V} \cdot d\vec{a}_6 = 1$$

Have students do this!
Idea: Try, look @ Gauss close book. General \rightarrow do on own!

so indeed ...

$$\int \nabla \cdot \vec{v} d\gamma = \oint \vec{v} \cdot d\vec{\alpha}$$

$$2 = 2$$

So what do we have?

$$1.) \vec{V} = x\hat{x} + 0\hat{y} = \leftarrow \quad \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array}$$

$$\vec{\nabla} \cdot \vec{V} = 1$$

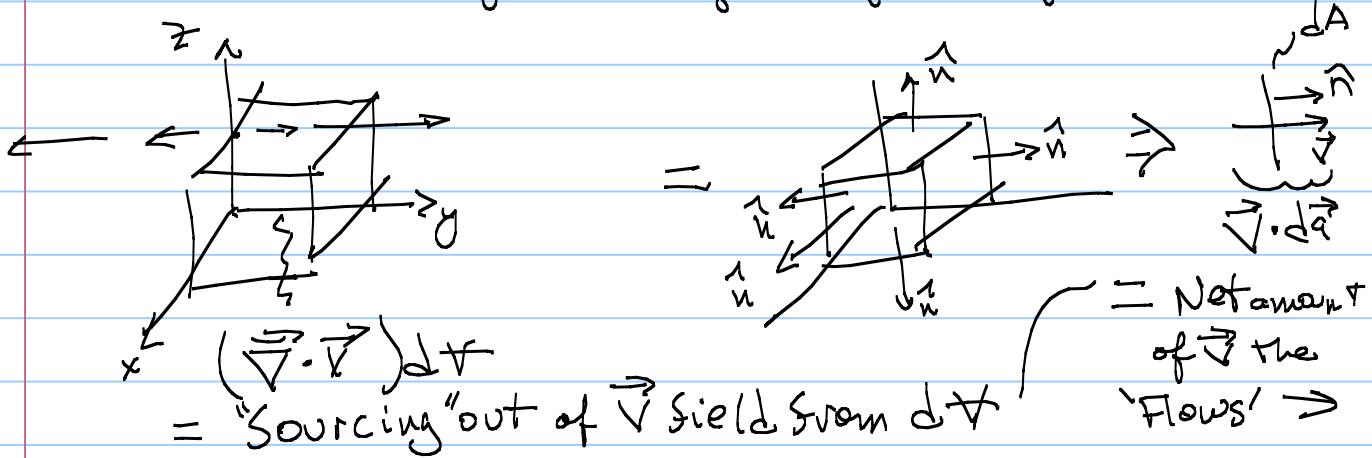
idea: $\vec{\nabla} \cdot \vec{V}$ = idea of how much vector fields spreads out from place to place or "Sources" out from place-to-place

2.) Now we have

Gauss's
Greens } Theorem
Div }

$$\oint_{\mathcal{F}} (\vec{\nabla} \cdot \vec{V}) dA = \underset{\substack{\text{Surface} \\ \text{of } \mathcal{F}}}{\oint} \vec{V} \cdot d\vec{A}$$

$$\text{ex: } \vec{V} = y^2\hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$$



out of ΔA

ie

$$\oint_{\Delta A} (\vec{\nabla} \cdot \vec{V}) dA = \int_S \vec{V} \cdot d\vec{A} \quad \text{so can look at}$$

(1) # of sources or (2) how much of
of \vec{V} in ΔA $\vec{\nabla} \cdot \vec{V} = \frac{\text{sources of } \vec{V}}{\text{volume}}$ \vec{V} flows out! thru S_s around ΔA

FLUID analogy! if $\vec{V} \Rightarrow$ flow of fluid $\dot{m} = g \vec{V}$
(incompressible)

$$\text{so } g \vec{V} = \frac{kg}{m^3} \frac{m}{s} = \frac{\text{mass}}{(\text{area})(\text{time})} = \text{Flow}$$

$$\int_S (g) \vec{V} \cdot d\vec{A} = \frac{kg}{m^3} \frac{m}{s} m^2 = \frac{kg}{s}$$

Then this is called Flux!

$= \frac{\text{mass of fluid}}{\text{time}} \text{ out of } dA$

$$\int_{\Delta A} (\vec{\nabla} \cdot (g \vec{V})) dA$$

of sources of the fluid looks like

Therefore

its a mass flux

Now Max's Eqn #1

$$\vec{\nabla} \cdot \vec{E} = \frac{g}{\epsilon_0} \quad \text{where} \quad g = \frac{\text{charge}}{\text{volume}}$$

w/ this interpretation

\vec{E} \propto Electric Field

$$\therefore \vec{\nabla} \cdot \vec{E} = \frac{\# \text{ of sources of } \vec{E}}{\text{Volume}} = \frac{1}{\epsilon_0} g \left(\frac{\text{charges}}{\text{volume}} \right)$$

so charges = pt sources

So

\vec{E} fields that

DIVERGE!

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow$ no pt sources for \vec{B} fields
that DIVERGE!

$$\int_F (\vec{V} \cdot \vec{V}) d\tau = \oint_A \vec{V} \cdot d\vec{A}$$

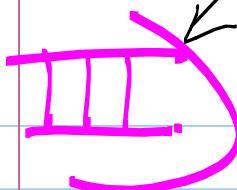
Homework: Griss p 1.32

Test Div Theorem for

$$\vec{V} = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$$

over \mathcal{V} = (cube volume, side=2)

3rd Vector deriv ($\vec{\nabla} \times \vec{V}$) Fund Theorem



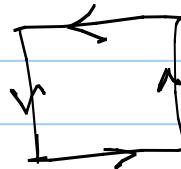
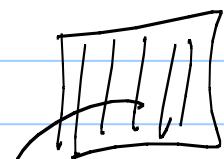
Fundamental Theorem of Curls

Recall: just studying $\vec{V} = \frac{\vec{F}}{|\vec{F}|}$, $\vec{\nabla} \times \vec{V} = \omega$ to measure

$$\oint_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{a} = \oint_{\text{path}} \vec{V} \cdot d\vec{l}$$

Again

$$S = \oint_{\text{perimeter}} \text{of surface}$$



surface

perimeter!

fluids inherently most have $\vec{\nabla} \times$ curls

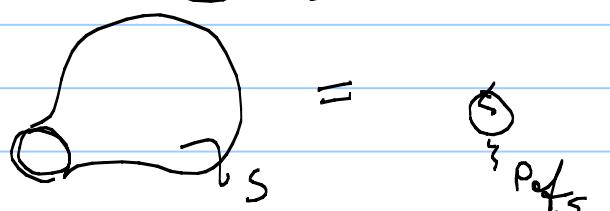


i.e. slow over wings so true fluids \neq Bernoulli But Navier-Stokes

highly non linear,

Again by explicit example

$\circlearrowleft r$



$$\int_S (\vec{V} \times \vec{U}) \cdot d\vec{a} = \oint_{\text{P of } S} \vec{V} \cdot d\vec{U} \quad \text{by example 1st}$$

$$\oint (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint \vec{v} \cdot d\vec{e}$$

↓
here goes!

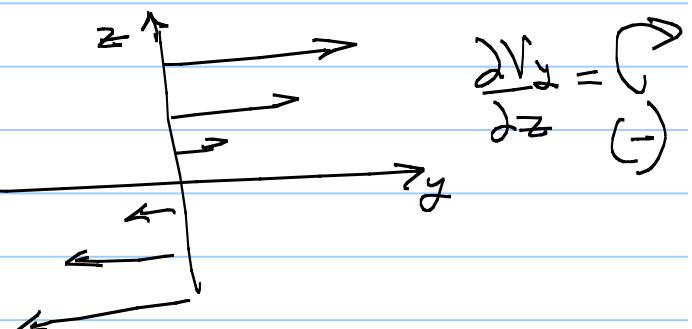
$$= \left(\frac{\partial v_z - \partial v_y}{\partial y - \partial z} \right) \hat{i} + \left(\frac{\partial v_x - \partial v_z}{\partial z - \partial x} \right) \hat{j} + \left(\frac{\partial v_y - \partial v_x}{\partial x - \partial y} \right) \hat{k}$$

(concentrate on just \hat{x}, \hat{y})

$$\left(\frac{\partial v_z - \partial v_y}{\partial y - \partial z} \right) \hat{i} = \begin{cases} z & \uparrow \\ y & \downarrow \end{cases} \quad \frac{\partial v_z}{\partial y} = \begin{cases} \circlearrowleft & (+) \\ \circlearrowright & (-) \end{cases}$$

means $\hat{z} = \hat{i}$

$$= \hat{z} = +\hat{i}$$



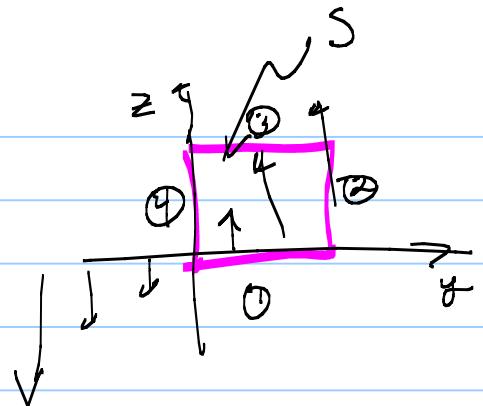
Now

$$da = da \hat{i} = dy dz \hat{i}$$

So for simplicity lets say

$$\vec{V} = 0\hat{i} + 0\hat{j} + y\hat{z} = V_x\hat{i} + V_y\hat{j} + V_z\hat{z}$$

Now



$$\int_S (\vec{V} \times \vec{V}) \cdot d\vec{a} = \int_S \vec{V} \cdot d\vec{l}$$

need to specify!

$$\int_S \left(\frac{\partial V_z}{\partial y} \right) \hat{i} \cdot d\vec{a} = \int_S \vec{V} \cdot d\vec{l} = \int_S \vec{V} \cdot d\vec{l}_x + \int_S \vec{V} \cdot d\vec{l}_y + \int_S \vec{V} \cdot d\vec{l}_z$$

$dy dz \hat{x}$
by RHR

since $\vec{V} = y\hat{z}$

$$\int_S \left(\frac{\partial V_z}{\partial y} \right) dy dz =$$

only dy & dz count

$$d\vec{l}_z = dz \hat{z}$$

$$d\vec{l}_y = dz \hat{z}$$

$$= \int_0^1 y dz + \int_{z_1}^{z_2} dz \hat{z}$$

$$\boxed{\int_S \left(\frac{\partial V_z}{\partial y} \right) dy} \int_S (1) dy dz$$

holding
 x, z const

$= V_z$

y_1, z_1

y_2, z_2

$\int_{z_1}^{z_2} y dz$

$$= \int y dz$$

$$\oint_S (\vec{V} \times \vec{V}) \cdot d\vec{a} = \oint_S \vec{V} \cdot d\vec{a} = \oint_S (v_x \hat{i} + v_y \hat{j} + v_z \hat{k}) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})$$

$$\int \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{i} \cdot (dy dz) \hat{i} = \int_v v_x(x, y, z) dx +$$

$$\int \frac{\partial v_z}{\partial y} dy dz - \int \frac{\partial v_y}{\partial z} dy dz$$

$$\int_y v_y(x, y, z) dy +$$

$$\int_z v_z(x, y, z) dz$$

Now add the others

$$\int \frac{\partial v_x}{\partial z} dx dz - \int \frac{\partial v_z}{\partial x} dx dz$$

$$\int \frac{\partial v_y}{\partial x} dx dy - \int \frac{\partial v_x}{\partial y} dy dx$$