

E.F. Donevay / BSC Phys Ph438:

Note Title

Ordinary Derivs

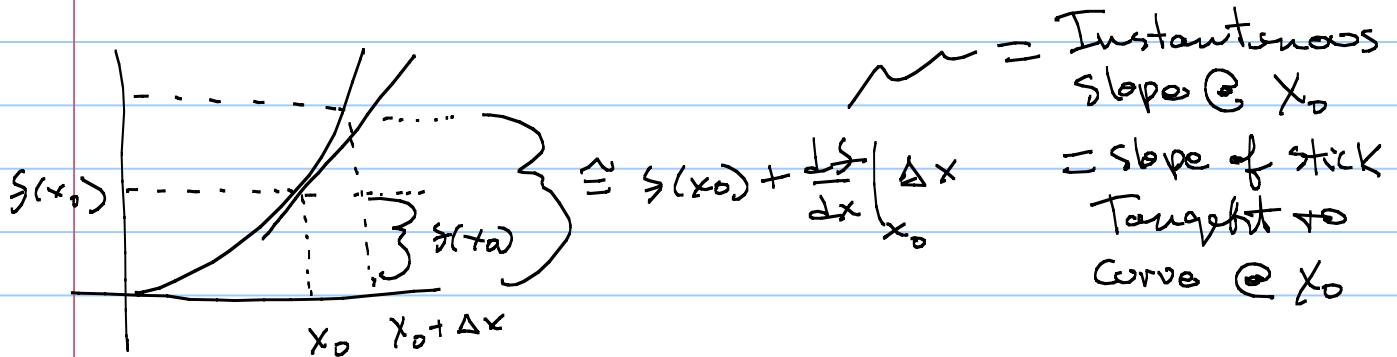
Partials to

$\hat{d}x$ & \hat{G}

22/2004

Griffiths HW 1.12

Start: Ordinary 1-D deriv



in limit of $\Delta x \rightarrow dx$ it is exact

so

$$f(x_0 + dx) = f(x_0) + \frac{df}{dx} \Big|_{x_0} dx$$

or

$$df = f(x_0 + dx) - f(x_0) = \frac{df}{dx} \Big|_{x_0} dx = f'(x)$$

$$df = \left(\frac{df}{dx} \right) dx$$

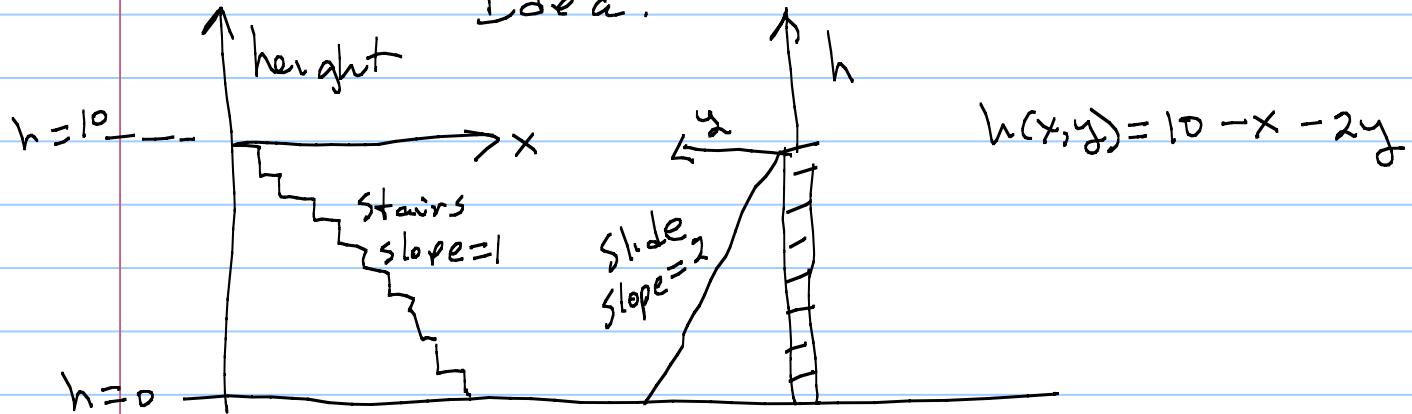
WORDS!

• is go from, change, x by amount dx , Then $f(x)$ changes by the amount df .

$$\therefore \left(\frac{df}{dx} \right) = \text{slope} = \frac{\text{instant amount of change in } f}{\text{unit length of } x}$$

Very Cool... can we extend ordinary 1-D
derivs to Partial derivatives?

Idea:



So hill height = $S(x,y)$

If all you cared about was x Then could say

$$\text{slope in } x = \left. \frac{\partial S(x,y)}{\partial x} \right|_y = \text{partial of } S(x,y)$$

w/ respect to x HOLDING all other variable constant!

* Another way to say this: partial derivative = deriv w/ respect to EXPLICIT variable while holding all others constant!

Likewise is just wanted to know slope in y

$$\text{Recall: } \ddot{y} = \frac{1}{2}m\dot{y}^2 - mgy \left(\frac{\partial \dot{y}}{\partial t} \right) - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = 0$$

$$= \left. \frac{\partial S(x,y)}{\partial y} \right|_x = \text{partial } S(x,y) \text{ w/r to } y \\ \text{Holding all other variables constant}$$

So

$$\text{Clearly Slope } x = \left. \frac{\partial S}{\partial x} \right|_{y=\text{const}} = -1$$

$$\text{Slope } y = \left. \frac{\partial S}{\partial y} \right|_{x=\text{const}} \quad \left. \begin{array}{l} \text{so } \frac{\partial S}{\partial x} = 0 \\ \text{so } \frac{\partial S}{\partial y} = 0 \end{array} \right\} h(2,0) = 10 - 2 - 0 = 8$$

$$\left. \begin{array}{l} \\ \end{array} \right\} h(0,2) = 10 - 0 - 4 = 6$$

lets Formalize things a bit

in 1-D

$$ds = \left(\frac{ds}{dx} \right) dx$$

words

go from x amount dx
is $s(x)$ changes by ds

in 2-D like our stairs ----

is $dy=0$, only change in dx

$$dh = \left(\frac{dh}{dx} \right) dx$$

or

if $dx=0$, only change in dy

$$dh = \left(\frac{dh}{dy} \right) dy$$

But in general I am free to move $dx \neq dy$
@ will !

so

$$dh = \underbrace{\left(\frac{\partial h}{\partial x} \right) dx}_{\text{now call}} + \underbrace{\left(\frac{\partial h}{\partial y} \right) dy}_{\text{the partial}} = \text{total deriv}$$

Now call
'the partial'
change in dh → just
changes
 $\frac{\partial h}{\partial x} dx$ due to just x changes,
ie the partial
deriv of w/r to x

"

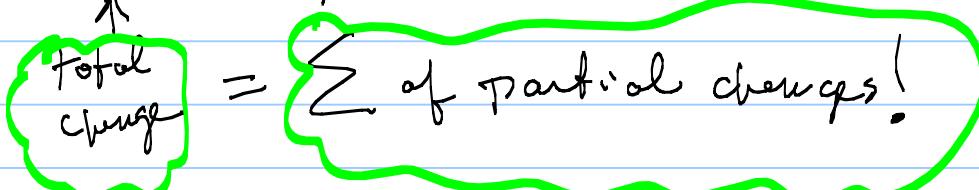
"

$\frac{\partial h}{\partial y} dy$

just
changes
ie

of h w/r to y

$$\text{So } dh = \delta h_x + \delta h_y$$



$$= \sum \text{ of Partial changes!}$$

Again
total changes They

$$dh = \underbrace{\left(\frac{\partial h}{\partial x} \right) dx + \left(\frac{\partial h}{\partial y} \right) dy + \left(\frac{\partial h}{\partial z} \right) dz + \left(\frac{\partial h}{\partial t} \right) dt}_{\sum \text{ of Partial changes!}} \dots$$

* Keep in mind $dx, dy, dz, dt = \xrightarrow{\text{to be what they want to be}}$

lets keep it to spacial changes

$$dh = \left(\frac{\partial h}{\partial x} \right) dx + \left(\frac{\partial h}{\partial y} \right) dy + \left(\frac{\partial h}{\partial z} \right) dz$$

$$dh = \delta h_x + \delta h_y + \delta h_z$$

this total change is suggestive of more interesting physical math! let's see

dh , stick to Grass

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

$$= \left(\frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right) \cdot \underbrace{(dx \hat{x} + dy \hat{y} + dz \hat{z})}$$

why not

$$= \text{introduce } \vec{\nabla}$$

$$(\vec{\nabla} T) \cdot d\vec{r}$$

$d\vec{r} \neq d\vec{r}$
keeping w/
Grass

which \Rightarrow 's

= vector But
it

$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z}$$

such that (ie vector $\vec{\nabla}$)

is nothing
until it
'operates'
on something
which must
be a scalar
cause not

$$\vec{\nabla} T = \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z}$$

No big deal $dT = \underset{\text{in } T \text{ by moving } d\vec{r}}{\text{Total change}}$ $\Rightarrow \vec{\nabla} T \cdot d\vec{r}$ dot or cross

so, here

we are: $dT = \text{Total change} \rightarrow = \vec{\nabla} T \cdot d\vec{r}$
in T by moving $d\vec{r}$

call $\vec{\nabla} = "del"$ vector operator ($\vec{\delta}$)

say $\vec{\nabla}$
 $"dT = \vec{\nabla} T \cdot d\vec{r}"$

So helps us to find dT , is $\vec{\nabla}$ useful
any other ways?

Answer is YES!

$\vec{\nabla}$ plays central role in 3
Types of vector derivatives!

No surprise

in 1-D

$$\frac{d}{dx} = \text{Deriv}$$

only 1
Flavor of
deriv!

3 flavors
of
vector
deriv.

w

in 3-D Vectors

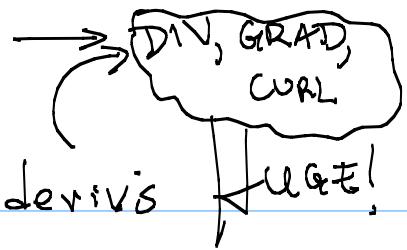
$\vec{\nabla}$ is a vector so expect to
be able to

1) $\vec{\nabla} \cdot \text{scalar} : \vec{\nabla} T = \text{GRADIENT}$

2) $\vec{\nabla} \cdot \text{vector} : \vec{\nabla} \cdot \vec{V} = \text{Divergence}$

3) $\vec{\nabla} \times \text{vector} : \vec{\nabla} \times \vec{V} = \text{GURL}$

ref:



$\vec{\nabla}$ leads to 3 flavors of vector derivis \downarrow URG!

start w/ $\vec{\nabla} T = \det \vec{e}_i^1$ scalar \equiv GRADIENT!

Gradient

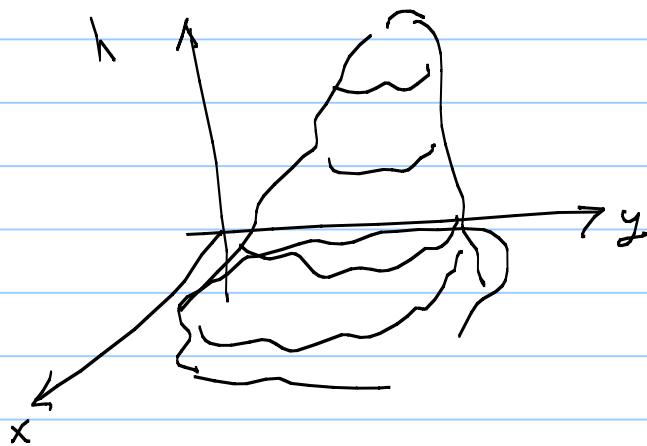
$$\delta T = \vec{\nabla} T \cdot d\vec{l} = \text{Grad } T \cdot d\vec{l}$$

↑
total change in
 δT by
moving $d\vec{l}$

lets see if $\text{Grad } T$ is something Physical!

OK:

$h(x, y) = \text{height} \Rightarrow$ now think of mountain



Now $dh = \vec{\nabla} h(x,y) \cdot d\vec{l} =$ total change in height from moving
 $d\vec{l} = dx\hat{i} + dy\hat{j}$

$$dh = |\vec{\nabla} h| |\vec{l}| \cos \theta$$

$\vec{\nabla} h$
 $d\vec{l}$
↑ See to be anything, let's just say $|d\vec{l}| = 1$ and then we'll go & around w/ its direction to see what happens.

1st thing to notice is that

$$dh = \max \text{ (ie max change)} \text{ is } d\vec{l} \text{ is } \parallel \vec{\nabla} h \text{ ie } d\vec{l} \theta = 0$$

if $d\vec{l}$ is any other direction, then dh is less
so conclude (1&2)

- IDEAS TO THINK
{ 1.) $\vec{\nabla} h = \vec{\text{grad}} h$ 'points' (ie is direction)
of the maximum increase in h
- 2.) $|\vec{\nabla} h| = \text{mag of } \vec{\text{grad}} h = \text{Slope (rate of } f)$
of h along that max direction
IE: MAX SPACE RATE OF CHANGE of h

Got it! $\vec{\nabla}h$ = vector deriv of scalar \Rightarrow pts in dir of 'max' change in h
 \Rightarrow mag is the 'max' slope

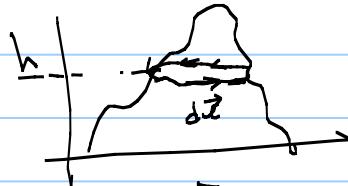
FURTHER --- Physical idea!
 Ok now think of this

$$dh = \vec{\nabla}h \cdot d\vec{r}$$

So ~~if~~ $\vec{\nabla}h \cdot d\vec{r} = 0$
 $= |\vec{\nabla}h| |d\vec{r}| \cos \theta = 0$

\Rightarrow 's

$dh = 0$ ie no change in height so $d\vec{r}$ must stay along 'CONTOUR' of the same height (ie contour map)



Since $d\vec{r}$ moves all the way around this 'contour' $|d\vec{r}| \neq 0$
 & clearly $|\vec{\nabla}h| \neq 0$ (ie height does change)

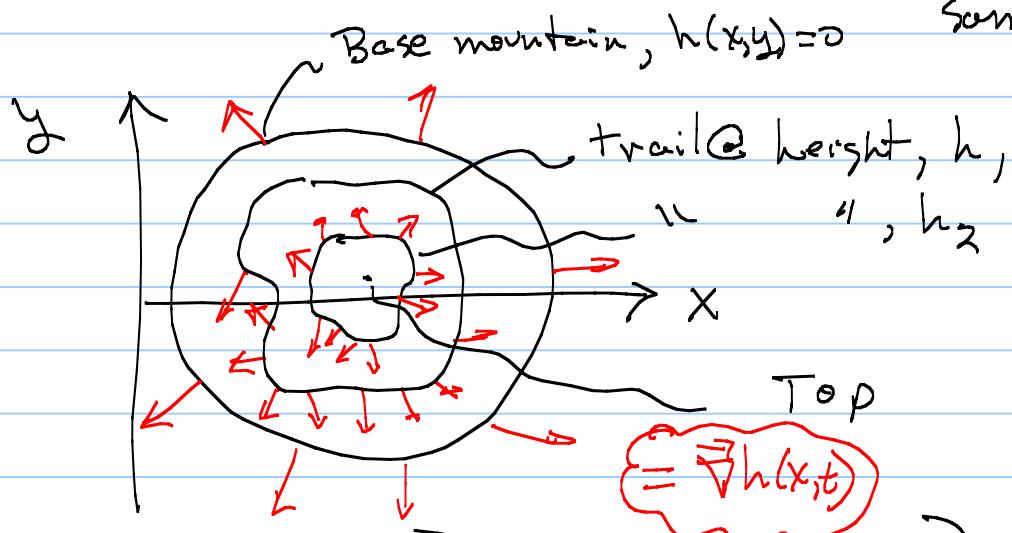
So ... in this case $\theta = 90^\circ$

which means that

$\nabla h = 0 \Rightarrow \nabla \vec{h}$ stays on contours
and

∇h must then be \perp to $\nabla \vec{h}$ or \perp to 'contours' (ie constant h 's)

i.e. Contour map of mountain \equiv lines follow path of some height!



So what does $\nabla h(x,y)$ look like?

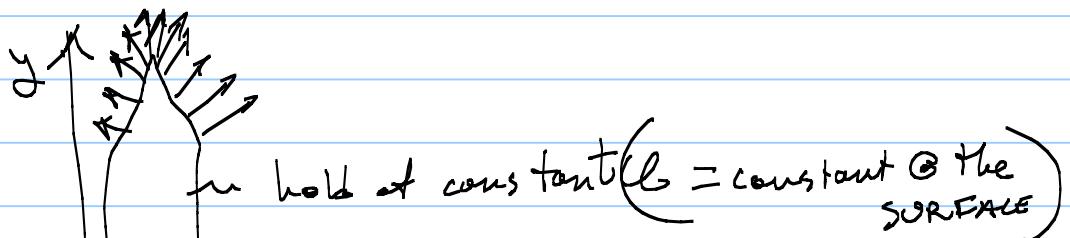
Answer \perp everywhere to these contour lines!

$\nabla h(x,t) = \perp$ to constant h
everywhere

This is HUGE!

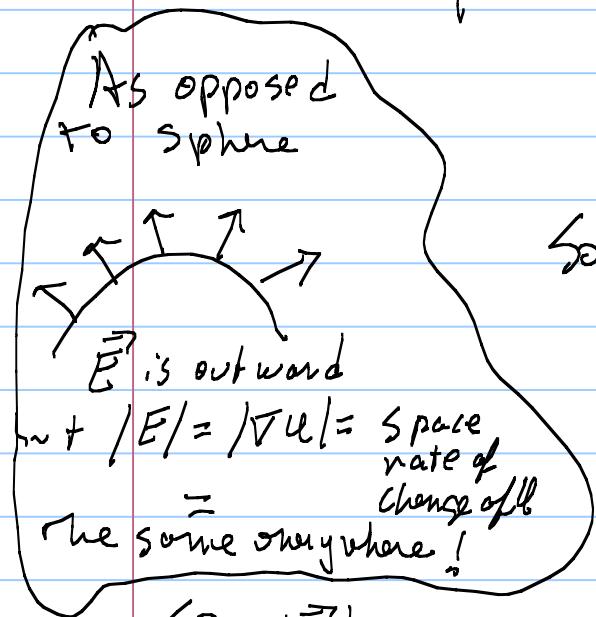
If ever you see a SCALAR that is Constant, Then
→ Scalars will pt \perp to that scalar surface!

ex: Lightning Rod: ($\vec{E} = -\nabla \phi$
 \uparrow electric potential)



$$\phi(x, y) = S(x, y) = \text{constant}$$

describes
the surface
of the lightning



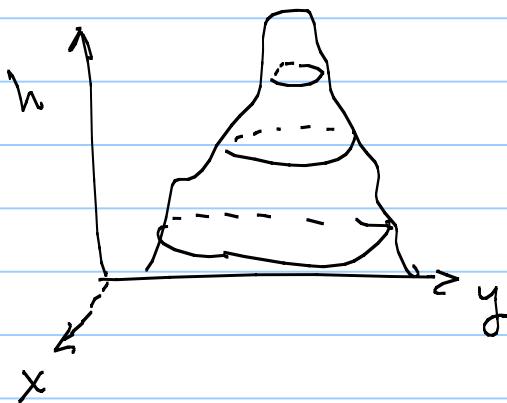
\vec{E} is outward
 $\nabla \phi = |E| = \text{space rate of change of } \phi$
the same anywhere!

1.) $\vec{\nabla} \phi = \text{every } \perp \text{ to the surface}$

2.) $|\vec{\nabla} \phi| = \text{max}$ (can see @ tip)
space rate of change of surface

So $|E| = \text{max } @ \text{tip}$ = where $|E|$ is enough to ionize gas (air) so that charges flow = lightning!

Finally!



Again

$$dh = \vec{\nabla} h \cdot d\vec{l}$$

we look @

$$dh = 0$$

For $d\vec{l}$ stayed on
a contour line.

Now let $d\vec{l}$ be completely
free again

$$\nabla dh = \vec{\nabla} h \cdot d\vec{l} = 0$$

$$\left(\frac{\partial h}{\partial x} \hat{x} + \frac{\partial h}{\partial y} \hat{y} \right) \cdot \underbrace{\left(dx \hat{x} + dy \hat{y} \right)}_{\text{For } dx \neq 0, dy \neq 0} = 0$$

For $dx \neq 0$
 $dy \neq 0$

Then

$$\left. \begin{aligned} \frac{\partial h}{\partial x} &= 0 \\ \frac{\partial h}{\partial y} &= 0 \end{aligned} \right\}$$

just as \min

$$\frac{\partial f}{\partial x} = 0 = \begin{cases} \text{---} & \text{min} \\ \text{---} & \text{or max} \end{cases}$$

$$\frac{\partial h}{\partial x} = 0 \quad = 2-D \text{ min or max}$$

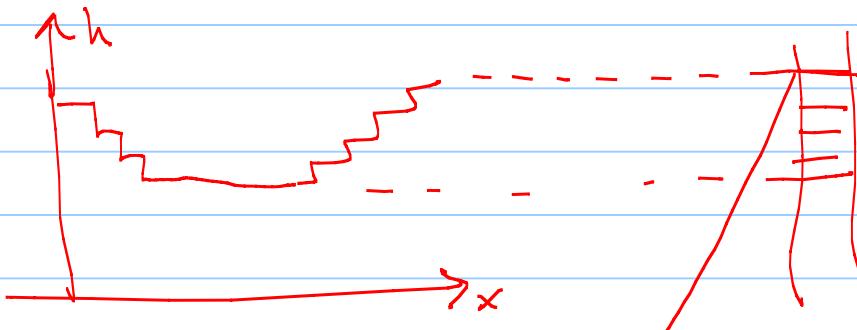
$$\frac{\partial h}{\partial y} = 0$$

so solve for x that satisfies $\frac{\partial h}{\partial x} = 0$, say x_0

they have to solve simultaneously
the simul

so that $\frac{\partial h}{\partial x}|_{x_0, y_0} = \frac{\partial h}{\partial y}|_{x_0, y_0} = 0$ otherwise

Any you will have found both the x_0 & y_0



could have
a max or
min in x
& y w/o
the other!

Grid: 1.12 height of hill

$$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$$

plot this?

plot contours?

* h = in feet!
 x = miles east
 y = miles north

In any case Top of The hill Location

$$dh = \nabla h \cdot d\vec{r} = 0 ; (d\vec{r} \neq 0, \text{ ie specifically } dx, dy \neq 0)$$

$$\frac{\partial h}{\partial x} \stackrel{x}{=} \frac{\partial h}{\partial y} \stackrel{y}{=} 0 \quad \text{simultaneously!}$$

$$\nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right) = 10(2y - 6x + 0 - 18 + 0 + 0) = 0$$

$$\begin{aligned} \frac{\partial h}{\partial x} &= 10(2y - 6x + 0 - 18 + 0 + 0) = 0 \\ \frac{\partial h}{\partial y} &= 10(2x - 0 - 8y + 0 + 28 + 0) = 0 \end{aligned}$$

$$\left. \begin{aligned} -6x + 2y &= 18 \\ 2x - 8y &= 28 \end{aligned} \right\} \begin{array}{l} \text{simultaneous} \\ \text{linear equals} \\ \text{for 2 unknowns!} \end{array}$$

$$\begin{array}{l} -6x + 2y = 18 \\ 2x - 8y = -28 \end{array} \quad \left. \begin{array}{l} \\ \end{array} \right\}$$
 look just 2 simlt equations
 let get 'nicer'
 w/ little linear algebr
 an row reduction

$\begin{matrix} 3 & * R_2 \end{matrix}$

$$\begin{array}{l} -6x + 2y = 18 \\ 6x - 24y = -84 \end{array}$$

add R_1 to R_2

$$\begin{array}{l} -6x + 2y = 18 \\ 0 - 22y = -66 \\ y = 3 \end{array}$$

is $y = 3$

Then plug back in

$$x = \frac{1}{4} \begin{vmatrix} 18 & 2 \\ -28 & -8 \end{vmatrix} = \frac{-144 + 56}{48 - 4} = \frac{-88}{44} = -2$$

$$y = \frac{1}{4} \begin{vmatrix} -6 & 18 \\ 2 & -28 \end{vmatrix} = \frac{(-6)(-28) - (18)(2)}{44} = \frac{168 - 36}{44} = \frac{112}{44} = 3$$

$$-6x + 2(3) = 18$$

$$-6x = 12$$

$$x = -2$$

a.) Top 2

so $x_0 = -2, y_0 = 3$ simultaneously get $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0 = \text{Max}$

b.) $\Rightarrow h(x_0, y_0) = 20(-6) - 30(4) - 40(2) - 180(-2) + 280(3) + 120 = 720$

C. how steep, $|\vec{\nabla}h|$, in ft/mile

@ 1 mile North : $y = +1$

1 mile East : $x = +1$

$$\vec{\nabla}h(x,y) = (20y - 60x - 180)\vec{x} + (20x - 80y + 280)\vec{y}$$

From

$$\vec{\nabla}h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right) = 10(2y - 6x + 0 - 18 + 0 + 0) = 0$$

$$\frac{\partial h}{\partial x} = 10(2x - 0 - 8y + 0 + 28 + 0) = 0$$

$$\vec{\nabla}h(1,1) = (20 - 60 - 180)\vec{x} + (20 - 80 + 280)\vec{y}$$

$$= -220\vec{x} + 220\vec{y}$$

$$\therefore |\vec{\nabla}h(1,1)| = \sqrt{2}(220) \approx 308 \frac{\text{ft}}{\text{mile}}$$

