

E.F. Deveney / BSC Physics: Series Expansions

Note Title

→ Taylor
→ Mac
Linear Algebra
V.S. vs F.S
→ Binomial
→ Fourier

Idea: if know Everything about a function, say $s(x)$, at 1 pt, say x_0 , ONLY! ie $s(x_0)$

$$\frac{ds}{dx}\Big|_{x_0}, \frac{d^2s}{dx^2}\Big|_{x_0} + \dots$$

Can you 'recreate' entire $s(x)$, ie at all other x , from the info just @ x_0 ?

The Answer is **YAR!**

∞ - Series Expansion:

Why? General idea

V.S.

\vec{F} = expansion onto 'complete' vectors Basis

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

F.S.

function spaces have only dimensions Basis that form a 'complete' basis in Function Space!

* Some F.S. have all the properties of V.S. including dot product. These are called Hilbert Space

Hilbert was huge in Q.M. & his student was
Emmy Noether of 'Noethers Theorem'

A good & complete F.S. is, one, polynomials
of order n : which are linearly indep & 'complete'

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} \xrightarrow{\text{V.S.}} S(x) = q_0 x^0 + q_1 x^1 + q_2 x^2 + \dots + q_n x^n$$

* Note: Careful that
actually converge
(ie $x < 1$)

F_x, F_y, F_z = "projections"
of \vec{F} onto vector basis

* $F_x = \hat{i} \cdot \vec{F}$ by
"dot product"

1) x^n 's, $n=0, 1, 2, \dots$
good basis (linearly indep & complete)

But not Hilbert!

Legendre Polynomials
= built from

x^n 's are good

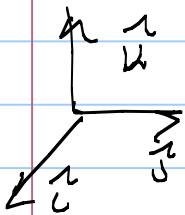
Hilbert space used
in Q.M.

Ans = projections of $S(x)$ onto function
basis & given by the equivalent
representation of the dot product in F.S. $a_n = S(x) x^n dx$

Complete & linearly independent in V.S.

(versus)

Function Space



V.S. $\hat{u}, \hat{v}, \hat{w}$

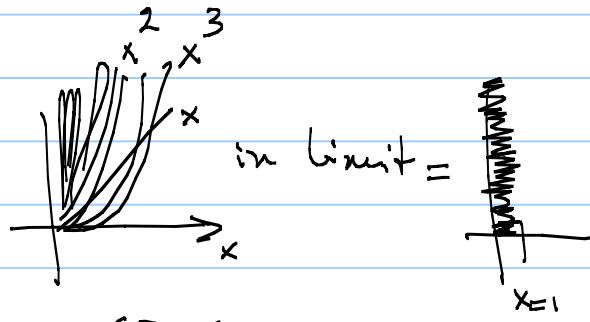
"span"

The
entire
space

So can build
any \vec{r} in 3D

$\vec{r} = F_x \hat{u} + F_y \hat{v} : F_x, F_y = \text{"projections"}$
of \vec{r} onto
vector basis

V.S.
 $\hat{u}, \hat{v}, \hat{w} =$
linearly independent
& complete!



\Rightarrow So

$|x| \leq 1, x^n \text{ "fill"}$
the

function space w/o
holes!

so can build any

$s(x) = a_0 x^0 + a_1 x^1 + \dots$ a_n = "projections"
of $s(x)$ onto function
basis

For
V.S.

det Wronskian $\neq 0$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{array}{l} \text{row} \\ \text{echelon} \\ \text{form} \end{array}$$

$\vec{B} - C\vec{A} \neq 0$
for all C
i.e. \vec{A} is
or \vec{B} not
dimensions
of space

So clearly $\hat{i} - C\hat{j} \neq 0$ i.e. linearly independent.

ex:

$\sin(x)$ & $\cos(x)$

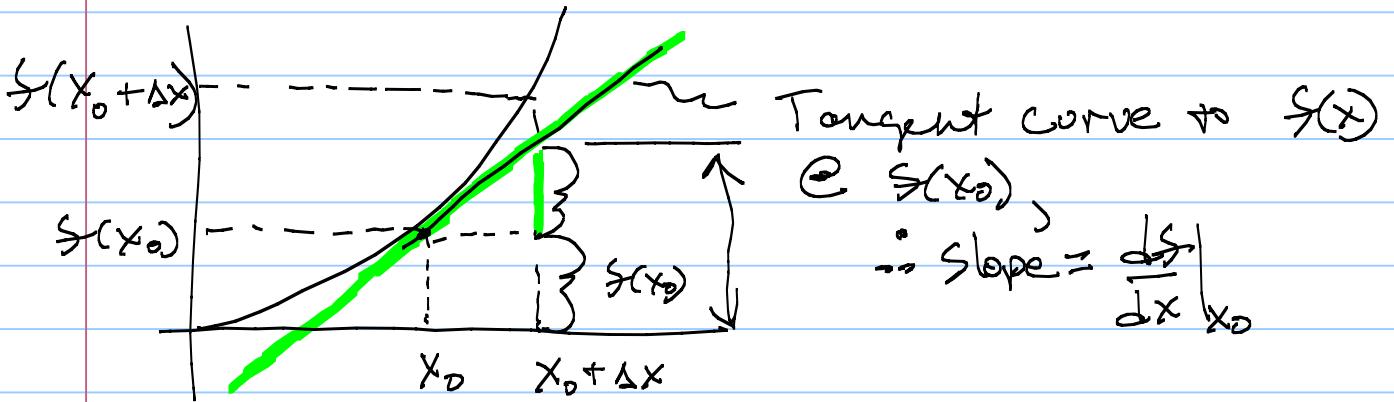
Never have

$$\sin(x) - C \cos(x) = 0$$

$\therefore \sin(x)$

& $\cos(x) =$
linearly independent

Geometric Argument



↑

we
know all
 $s(x_0)$

$\left. \frac{ds}{dx} \right|_{x_0} \equiv s'(x_0)$

$\left. \frac{d^2s}{dx^2} \right|_{x_0} \equiv s''(x_0)$

Can we get
 $s(x_0 + \Delta x)$?

So can estimate

$\begin{array}{c} \approx y = mx + b \\ \rightarrow \Delta x \leftarrow \end{array}$
 $= \left. \frac{ds}{dx} \right|_{x_0} \Delta x + b$

Can see

$$s(x_0 + \Delta x) \stackrel{\text{def}}{=} s(x_0) + \left. \frac{ds}{dx} \right|_{x_0} \Delta x$$

TAYLOR SERIES result

EXACT

$$s(x_0 + \Delta x) = s(x_0) + s'(x_0) \Delta x + \frac{1}{2} s''(x_0) \Delta x^2 + \frac{1}{3!} s'''(x_0) \Delta x^3 + \dots$$

↓ Es we have done it!

$f(x_0 + \Delta x) =$ Series expansion based on $f(x_0)$

* Note: if x_0 , the pt we know all about is

~~x_0~~ $x_0 = 0$

Then

$$f(x_0 + \Delta x)$$

$$f(0 + \Delta x)$$

$$f(\Delta x)$$

why not

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \dots$$

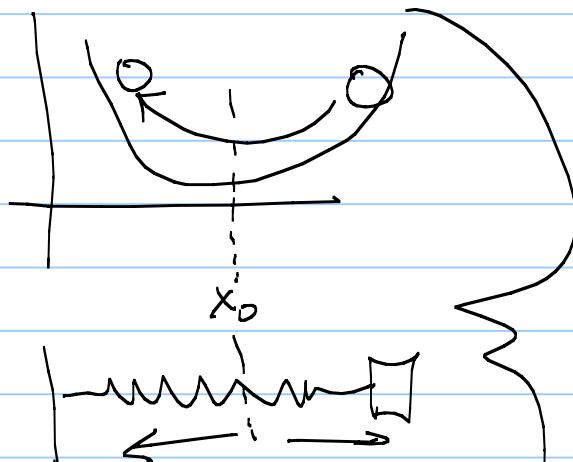
Special Name: MacLaurin Series!

In many problems in physics, it is often easier, or more convenient, to replace tough function w/ easier series expansion.

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \Delta x$$

$\Delta x \ll 0$ is small.

Ex: Near stable equi



looks like springs

$$\text{Energy} = \frac{1}{2} kx^2$$

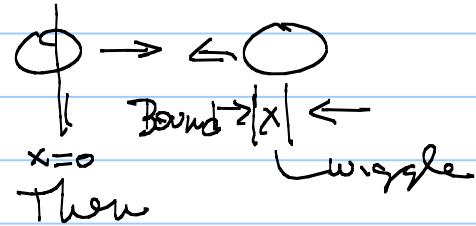
and

$$F = -kx$$

Linear restoring force

k = spring constant

If I know Energy of an atomic system @ equi, say atoms bound in a solid,

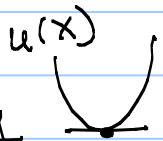


$$U(x_0 + \Delta x) = U(x_0) + U'(x_0) \Delta x + \frac{1}{2} U''(x_0) \Delta x^2$$

let $U(x_0) = 0$

since

equilib in energy



Slope @ equi = 0

$$\therefore U'(x_0) = 0$$

$$U(x) \approx \frac{1}{2} U''(x_0) x^2$$

so must be approx interatomic force that

resembles spring w/ $\vec{F} = -k\vec{x}$

∴

Atoms in solids \approx



wiggle!

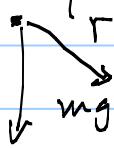
More tang examples of Taylor & Mac series expansion.

 Simple pendulum

$$\sum \vec{F} = I \ddot{\theta}$$

$$mg \vec{r} = I \frac{d\theta}{dt^2} \vec{r}$$

$$-l mg \sin \theta = I \frac{d^2\theta}{dt^2}$$



$$\frac{d^2\theta}{dt^2} + \frac{lmg}{I} \sin \theta = 0 ; I=ml^2$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0$$

* Note

$$\frac{d\theta}{dt} = \frac{\theta - \theta_0}{t} \Rightarrow$$

Non linear diffy-Q.

\vec{r} closed



∴ Tough to solve ...

may be give up for moment & restrict to

problem you can solve, meaning can you get the diffy-Q linear?

Note: For small angles θ , about $\theta_0 = 0$

$$\begin{aligned}\sin(\theta_0 + \theta) &\approx \sin\theta \stackrel{\text{w}}{=} \sin(\theta) + \left(\frac{d\sin\theta}{d\theta}\Big|_{\theta_0}\right)\theta \\ &= \theta + \cos\theta\Big|_{\theta_0} \theta \\ &= \theta + (1)\theta\end{aligned}$$

or

$$\sin\theta \approx \theta \quad \left. \begin{array}{l} \text{Radians!} \\ \text{check how true!} \end{array} \right\}$$

$$\therefore \frac{d^2\theta}{dt^2} + \frac{g}{\ell} \theta = 0 = \text{nice linear}\\ \text{dSg-Q w/ soln}$$

$$\theta(t) = \theta_0 \cos(\omega t + \delta)$$

$$\omega = \sqrt{\frac{g}{\ell}}$$

H.W. Try: $\cos(\theta) =$

A special Case of Taylor Series is The
Binomial Series Expansion = HUGE!

$$\text{BSE: } (1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} +$$

for
 $x \approx 1$

$$\frac{m(m-1)(m-2)}{3!}x^3 + \dots$$

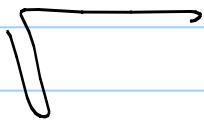
USED Throughout EM!

Ex: $\int \frac{\cos x \, dx}{(1+x)^7}$ yields!

For small x ... LOTS of terms, so not
so restrictive

$$\stackrel{\approx}{=} \int \cos x (1+7x) \, dx = \underbrace{\int \cos x \, dx + \int x \cos x \, dx}_{\text{NICE!}}$$

My Favorite trick!



in your



Head

$$\sqrt{70} = ?$$

$$\sqrt{64+6} = \sqrt{64\left(1 + \frac{6}{64}\right)} = 8\sqrt{\left(1 + \frac{6}{64}\right)}$$

$$= 8\left(1 + \frac{6}{64}\right)^{\frac{1}{2}}$$

use BSE $x = \frac{6}{64}$

$m = \frac{1}{2}$

$$= 8\left(1 + \frac{1}{2} \cdot \frac{6}{64}\right)$$

$$= 8 + \frac{1}{2} \cdot 8 \cdot \frac{6}{64}$$

$$\sqrt{70} = 8 + \frac{1}{2} \cdot \frac{6}{8}$$

$$\frac{3}{8} \approx .375$$

* $\sqrt{70} =$
 8.36660
 $= 8.37$

$$= 8.38$$

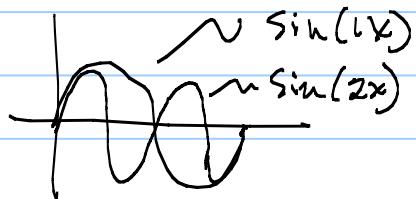
TRICKS that Simplify Form = $\frac{(\text{closest square})}{(\text{closest square})} + \frac{1}{2} \cdot \frac{(\text{distance})}{(\text{closest square})}$

Finally: $\sin(x), \cos(x)$ = linearly indep

$$\left\{ \begin{array}{l} \cos(x), \cos(2x), \cos(3x) \dots \\ \sin(x), \sin(2x), \sin(3x) \dots \end{array} \right.$$

are all linearly indep

ex:



could never get $\sin(x) - C \cdot \sin(2x) = 0$

So: $\sum_{n=0}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) =$ Complete
linearly
indep Basis

Set in
function

space



$$f(x) = \sum_{n=0}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) =$$

Fourier Series
expansion

a_n & b_n = projections!

I always write ~~expanding~~ by symmetry as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_x x) + b_n \sin(n\omega_x x)]$$

where

$$a_n \equiv \frac{1}{l} \int_{-l}^{+l} f(x) \cos(n\omega_x x) dx$$

$$b_n \equiv \frac{1}{l} \int_{-l}^{+l} f(x) \sin(n\omega_x x) dx$$

↓
another day