THE ONSET OF LINEAR INSTABILITIES IN A SOLID COMBUSTION MODEL

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This paper concerns the onset of linear instability in a simple model of solid combustion in a semi-infinite two-dimensional strip of width l. The free boundary problem that describes the model involves initial and boundary conditions, including a nonlinear kinetic condition at the interface. The linear problem governing perturbations to a basic solution is solved by the method of images with the reaction front perturbation satisfying an integro-differential equation. This equation is then solved using Laplace transforms. Finally, we perform a stability analysis for the model by studying the solution of the reaction front perturbation. The inclusion of initial conditions enables us to show the development of linear instability from arbitrary initial small disturbances.

1. Introduction

Using perturbation techniques, the method of images and Laplace transforms, this paper characterizes the onset of linear instability in a simple model of solid combustion. In gasless combustion, chemical reactions convert a solid fuel directly into solid products. The interplay between heat generation and heat diffusion in the medium controls the development of the flame front, while material diffusion can be ignored.

We consider here a version of the sharp-interface model of solid combustion introduced by Matkowsky & Sivashinsky (1978). Their free-boundary problem was posed on the whole real line for one-dimensional burning. We consider an interface propagating along an infinite strip of width l, taking only the region *ahead of the front* as the problem domain. This one-sided model was initially introduced by Frankel (1991) and can be considered to represent a solid combustion process whose burnt state has very low heat conductivity. Numerical simulations and analysis by

Frankel *et al.* (1994), (1995) demonstrate that the dynamics of the one- and two-sided models are virtually identical, suggesting that the heat transfer in the burned matter is qualitatively unimportant.

In addition to numerical work, asymptotic studies on this model (e.g., Gross (1997)) have distinguished between temporally periodic and non-periodic regimes and have identified dominant spatial patterns, valid for sufficiently large time. The current work considers *arbitrary* initial disturbances and captures the *onset* of linear instability, as initial conditions may play an important role in combustion. Some studies (for example, Gross (1997) and Volpert, *et al.* (1992)) have demonstrated that the evolution of stable patterns depends on initial conditions.

The geometry under consideration is a strip of material extending from $x = 0$ to $x = l$ and from $y = 0$ to $y = \infty$. Suppose the material is undergoing an exothermic reaction so that $y = f(t, x)$ defines the reaction front. The burned region is $y < f(t, x)$, and $y > f(t, x)$ is the unburned. See Figure 1. Using dimensionless variables, the temperature distribution $T(x, y, t)$ must satisfy the heat equation in the unburned region, i. e.,

$$T_t = T_{xx} + T_{yy}, \qquad 0 < x < l, \quad y > f(t, x), \quad t > 0. \tag{1.1}$$

At the interface position $\Gamma(t) = \{(x, y)|y = f(t, x)\}$, we impose two conditions

$$T|_\Gamma = 1 + \nu K(V) - \gamma \mathcal{K}, \qquad \left.\frac{\partial T}{\partial \mathbf{n}}\right|_\Gamma = -V. \tag{1.2a, b}$$

Here $V = f_t/\sqrt{1 + f_x^2}$ is the velocity of propagation of the front in the direction $\mathbf{n}$ normal to the front, ν is inversely proportional to the activation energy, $\mathcal{K}$ is the mean curvature of the interface and the coefficient γ is proportional to surface tension. Although $\gamma = 0$ for condensed state combustion that we consider here, the geometric effect of the interface is important in some physical settings such as the rapid solidification of overcooled liquids. The boundary kinetics $K(V)$ (given, for example, as a one-parameter family of functions: $K(V) = (V^p - V^{-0.7})/(p + 0.7)$ in Frankel & Roytburd (1994)) are normalized so that $K(1) = 0$ and $K'(1) = 1$. Impose zero-slope contact conditions for the interface

$$f_x(t, 0) = 0, \qquad f_x(t, l) = 0, \tag{1.3a, b}$$

and adiabatic (insulated) conditions at the walls

$$T_x(t,0)=0, \qquad T_x(t,l)=0. \tag{1.4a, b}$$

In accordance with the normalization of the temperature, we also require that

$$\lim_{y\to\infty} T(t,x,y)=0. \tag{1.5}$$

Finally, the initial temperature distribution is prescribed as

$$T(0,x,y)=g(x,y), \tag{1.6}$$

where $\lim_{y\to\infty} g(x,y)=0$ as required by (1.5). Equations (1.1)-(1.6) form a free-boundary, initial-value problem that defines the temperature T and the front position f uniquely. One may use (1.1), (1.2a), (1.4)-(1.6) to solve for the temperature T for any arbitrary front position f and then determine the front position by equations (1.2b) and (1.3).

The linear instability of the above system is analyzed by Gross (1997) using a normal mode method. The approach is well-suited to the study of large-time behaviors. Examining the early evolution from an arbitrary initial disturbance, however, would be difficult. This would require separating an arbitrary initial condition into unstable modes. In this paper, we instead use a procedure similar to that in Yu & Kevorkian (1992) and Yu *et al.* (1999). As our first step, we must understand the linearized problem. In Section 2, we solve the linearized problem for the perturbations to a basic solution by the method of images. The temperature distribution is expressed as an integral in terms of arbitrary initial perturbations. An integro-differential equation is derived for the reaction front perturbation. Laplace transform of the front perturbation is then solved and expressed as an integral in terms of arbitrary initial conditions. In Section 3, we invert the Laplace transform of the front perturbation for typical initial conditions. We consider two cases in order to complete our linear stability study. In Section 3.1, we obtain explicit solution of the reaction front perturbation for the flat mode case ($k=0$) and then perform a stability analysis to show the onset of linear instability for the model. In Section 3.2, we extend our method to include the non-flat mode case ($k\neq 0$) where the solution and the stability analysis can be carried out numerically. We show these results in the following sections.

2. The linearized solution; the method of images

We begin our discussion by studying the behavior of the linear problem governing perturbations to the basic solution of (1.1)-(1.6) consisting of a flat constant-speed front ($f(t,x)=t$, $T=e^{t-y}$). Note that the corresponding basic velocity is $V=1$. Proceeding as in Gross (1997), we first change the coordinates into the front-attached ones so that $\tau=t$, $\xi=x$ and $\eta=y-f(t,x)$. Perturbing about the basic solution by letting

$$T=e^{-\eta}+\epsilon w, \qquad f=\tau+\epsilon\phi \tag{2.1a, b}$$

and linearizing, one obtains the partial differential equation

$$w_\tau=w_{\xi\xi}+w_{\eta\eta}+w_\eta-e^{-\eta}(\phi_\tau-\phi_{\xi\xi}), \tag{2.2a}$$

subject to the linear boundary conditions

$$w|_{\eta=0}=\nu\phi_\tau-\gamma\phi_{\xi\xi}, \qquad w_\eta|_{\eta=0}=-\phi_\tau, \tag{2.2b, c}$$

$$w_\xi|_{\xi=0,l}=0, \qquad \phi_\xi|_{\xi=0,l}=0, \tag{2.2d, e}$$

$$\lim_{\eta\to\infty} w=0. \tag{2.2f}$$

Because of the homogenous boundary conditions (2.2d,e) on the ξ variable, w and ϕ can be expanded as cosine Fourier series in terms of eigenfunctions $\{\cos(k\xi), k=0,\frac{\pi}{l},\frac{2\pi}{l},...\}$. For each eigenmode with a fixed k ($k=0,\frac{\pi}{l},\frac{2\pi}{l},...$), we have

$$w(\tau,\eta,\xi)\sim W(\tau,\eta)\cos(k\xi), \qquad \phi(\tau,\xi)\sim\Phi(\tau)\cos(k\xi). \tag{2.3a, b}$$

The governing equation and boundary conditions for W and Φ are now given by

$$W_\tau=-\,k^2W+W_{\eta\eta}+W_\eta-e^{-\eta}(\Phi'+k^2\Phi) \tag{2.4a}$$

$$W|_{\eta=0}=\nu\Phi'+\gamma k^2\Phi, \qquad W_\eta|_{\eta=0}=-\Phi' \tag{2.4b, c}$$

$$\lim_{\eta\to\infty} W=0. \tag{2.4d}$$

The initial condition is

$$W(0,\eta)=g^*(\eta),\qquad \Phi(0)=\Phi_0, \tag{2.4e, f}$$

where Φ_0 is a constant and $g^*(\eta)$ is related to $g(x,y)$ in (1.6) through (2.3) and (2.1). Also, $\lim_{\eta\to\infty} g^*=0$ as requried by (2.4d). Equations (2.4a,b,d,e) define an initial and boundary value problem for $W(\tau,\eta)$ in a semi-infinite domain $\eta>0$, $\tau>0$ of the (η,τ) plane and conditions (2.4c,f) are for $\Phi(\tau)$ in the interval $\tau>0$.

In order to solve $W(\tau,\eta)$, we make a change of variable

$$u=(W-\nu\Phi'-\gamma k^2\Phi)e^{(k^2+\frac{1}{4})\tau+\eta/2}. \tag{2.5}$$

Substituting (2.5) into (2.4a,b,e) yields

$$u_\tau = u_{\eta\eta}+H(\tau,\eta) \tag{2.6a}$$

$$u|_{\eta=0}=0, \tag{2.6b}$$

$$u(0,\eta)=G(\eta), \tag{2.6c}$$

where $H(\tau,\eta)=-[\nu\Phi''+(\nu+\gamma)k^2\Phi'+k^4\gamma\Phi+e^{-\eta}(\Phi'+k^2\Phi)]e^{(k^2+\frac{1}{4})\tau+\eta/2}$ and $G(\eta)=[g^*(\eta)-\nu\Phi'(0)-\gamma k^2\Phi(0)]e^{\eta/2}$. Equation (2.4c) which is a condition for $\Phi(\tau)$, becomes

$$u_\eta|_{\eta=0}=-e^{(k^2+\frac{1}{4})\tau}\Phi'. \tag{2.7}$$

Using the method of images (see for example, Section 1.4 of Kevorkian (1990)), the solution of (2.6) can be written as

$$u(\tau,\eta)=\int_0^\tau dt\int_0^\infty \frac{H(t,x)}{\sqrt{4\pi(\tau-t)}}[e^{-(\eta-x)^2/4(\tau-t)}-e^{-(\eta+x)^2/4(\tau-t)}]dx$$
$$+\int_0^\infty \frac{G(x)}{\sqrt{4\pi\tau}}[e^{-(\eta-x)^2/4\tau}-e^{-(\eta+x)^2/4\tau}]dx. \tag{2.8}$$

Substituting the solution (2.8) for u into (2.7), we find the following condition for Φ:

$$\int_0^\tau dt\int_0^\infty \frac{xH(t,x)}{2\sqrt{\pi(\tau-t)^3}}e^{-x^2/4(\tau-t)}dx$$
$$+\int_0^\infty \frac{xG(x)}{2\sqrt{\pi\tau^3}}e^{-x^2/4\tau}dx=-e^{(k^2+\frac{1}{4})\tau}\Phi'(\tau). \tag{2.9}$$

Substituting the expressions for H and G into (2.9), one finds that Φ must satisfy

$$\int_0^\tau M(t)\int_0^\infty L(\tau-t,x)e^{x/2}dxdt+\int_0^\tau N(t)\int_0^\infty L(\tau-t,x)e^{-x/2}dxdt$$
$$+\int_0^\infty [g^*(x)-\nu\Phi'(0)-\gamma k^2\Phi(0)]e^{x/2}L(\tau,x)dx=-e^{(k^2+\frac{1}{4})\tau}\Phi'(\tau), \quad (2.10)$$

where $M(\tau)=-[\nu\Phi''+(\nu+\gamma)k^2\Phi'+k^4\gamma\Phi]e^{(k^2+\frac{1}{4})\tau}$, $N(\tau)=-(\Phi'+k^2\Phi)e^{(k^2+\frac{1}{4})\tau}$ and $L(\tau,x)=\frac{x}{2\sqrt{\pi\tau^3}}e^{-x^2/4\tau}$. Note that the first two integrations in t on the left hand side of (2.10) are convolution integrals involving a function defined by an integral in x with functions $M(\tau)$ and $N(\tau)$, respectively. Using $\tilde{M}$, $\tilde{N}$, $\tilde{\Phi}$ and $\tilde{L}$ to denote the Laplace transforms of M, N, Φ and L in τ, respectively, one finds that the Laplace transform of (2.10) yields

$$\tilde{M}(s)\int_0^\infty \tilde{L}(s,x)e^{x/2}dx+\tilde{N}(s)\int_0^\infty \tilde{L}(s,x)e^{-x/2}dx+\int_0^\infty [g^*(x)-\nu\Phi'(0)$$
$$-\gamma k^2\Phi(0)]\tilde{L}(s,x)e^{x/2}dx=-(s-\frac{1}{4}-k^2)\tilde{\Phi}(s-\frac{1}{4}-k^2)+\Phi(0), \quad (2.11)$$

where $\tilde{M}$, $\tilde{N}$ and $\tilde{L}$ are given as follows

$$\tilde{M}(s)=-[\nu(s-\frac{1}{4}-k^2)+\gamma k^2](s-\frac{1}{4})\tilde{\Phi}(s-\frac{1}{4}-k^2)$$
$$+[\nu(s-\frac{1}{4})+\gamma k^2]\Phi(0)+\nu\Phi'(0), \quad (2.12a)$$
$$\tilde{N}(s)=-(s-\frac{1}{4})\tilde{\Phi}(s-\frac{1}{4}-k^2)+\Phi(0), \quad (2.12b)$$
$$\tilde{L}(s,x)=e^{-x\sqrt{s}}. \quad (2.12c)$$

With $\tilde{L}$ defined in (2.12c), one can carry out all the integrals in (2.11) except the one involving the initial condition $g^*(\eta)$, and obtain

$$\frac{\tilde{M}(s)}{\sqrt{s}-\frac{1}{2}}+\frac{\tilde{N}(s)}{\sqrt{s}+\frac{1}{2}}+\int_0^\infty g^*(x)e^{-x(\sqrt{s}-\frac{1}{2})}dx-\frac{\nu\Phi'(0)+\gamma k^2\Phi(0)}{\sqrt{s}-\frac{1}{2}}$$
$$=-(s-\frac{1}{4}-k^2)\tilde{\Phi}(s-\frac{1}{4}-k^2)+\Phi(0). \quad (2.13)$$

Substituting $\tilde{M}$ and $\tilde{N}$ from (2.12) into (2.13) we can solve for $\tilde{\Phi}(s-\frac{1}{4}-k^2)$ in the form

$$\tilde{\Phi}(s-\frac{1}{4}-k^2) = \frac{1}{p(s)}\Big\{[1-\nu(\sqrt{s}+\frac{1}{2})-\frac{1}{\sqrt{s}+\frac{1}{2}}]\Phi(0) - \int_0^\infty g^*(x)e^{-x(\sqrt{s}-\frac{1}{2})}dx\Big\}, \tag{2.14}$$

where $p(s) = -\nu(\sqrt{s})^3+(1-\frac{\nu}{2})(\sqrt{s})^2+[\nu(\frac{1}{4}+k^2)-\gamma k^2-1]\sqrt{s}+\frac{1}{4}+\frac{\nu}{8}+(\frac{\nu}{2}-\frac{\gamma}{2}-1)k^2$.

Once an initial condition $g^*(\eta)$ has been specified, one can invert $\tilde{\Phi}$ in (2.14) to obtain the front perturbation Φ for any modes defined by constant k ($k \geq 0$). In the next section we perform stability analysis by solving the solution of the reaction front perturbation corresponding to typical initial conditions in the form

$$g^*(\eta) = g_0 e^{-\alpha\eta}, \tag{2.15}$$

where g_0 is a constant and $\alpha = \alpha_r + i\alpha_i$ is a complex number with $\alpha_r > 0$. Two cases will be illustrated. First, we study the special case with flat mode ($k = 0$) where explicit solution of the reaction front can be derived. Secondly, we show that numerical method can be used to solve the solution of the reaction front for all the cases with non-flat mode ($k \neq 0$) and as results, growth rates of the solution can be studied and the neutral stability curves can be obtained.

3. Solution of the front perturbation; stability analysis

With initial condition defined in (2.15) one can evaluate the integral in (2.14) and find that

$$\tilde{\Phi}(s-\frac{1}{4}-k^2) = \frac{1}{p(s)}\Big(\frac{-\nu s+(1-\nu)\sqrt{s}-\frac{1}{2}(1-\frac{\nu}{2})}{\sqrt{s}+\frac{1}{2}}\Phi(0) - \frac{g_0}{\sqrt{s}+\alpha-\frac{1}{2}}\Big), \tag{3.1}$$

where $p(s)$ is a third degree polynomial defined in (2.14).

3.1 <u>Flat mode case ($k = 0$):</u>

For the flat mode case ($k = 0$), (3.1) reduces to the following form:

$$\tilde{\Phi}(s-\frac{1}{4}) = \frac{\Phi(0)}{s-\frac{1}{4}} + q(s), \tag{3.2}$$

where $q(s)$ is given by

$$q(s) = \frac{1}{(\sqrt{s}+\alpha-\frac{1}{2})(\sqrt{s}-\frac{1}{2})[\nu s+(\nu-1)\sqrt{s}+\frac{1}{4}\nu+\frac{1}{2}]}. \tag{3.3}$$

Taking the inverse Laplace transform of (3.2) shows that

$$e^{\frac{\tau}{4}}\Phi(\tau) = e^{\frac{\tau}{4}}\Phi(0) + \mathcal{L}^{-1}\{q(s)\}. \tag{3.4}$$

In order to carry out the inverse Laplace transform we consider $q(s)$ as a quotient of polynomials in $\sqrt{s}$ so that a partial fraction representation for q can be obtained. Also, note that our partial fraction representation here allows complex numbers for roots and coefficients. The third factor in the denominator of $q(s)$ is quadratic in $\sqrt{s}$. Three cases result according to the discriminant of the quadratic:

1. if $\nu < \frac{1}{4}$, we have two real roots for the quadratic,

$$r_1 = -\frac{1}{2} + \frac{1+\sqrt{1-4\nu}}{2\nu}, \qquad r_2 = -\frac{1}{2} + \frac{1-\sqrt{1-4\nu}}{2\nu}. \tag{3.5a,b}$$

   A partial fraction expansion of $q(s)$ shows that

$$q(s) = \frac{A_1}{\sqrt{s}+\alpha-\frac{1}{2}} + \frac{A_2}{\sqrt{s}-\frac{1}{2}} + \frac{A_3}{\sqrt{s}-r_1} + \frac{A_4}{\sqrt{s}-r_2}, \tag{3.6}$$

   where A_1, A_2, A_3 and A_4 are constants.

2. if $\nu = \frac{1}{4}$, we have a repeated root, $r_1 = \frac{3}{2}$ for the quadratic and $q(s)$ becomes

$$q(s) = \frac{B_1}{\sqrt{s}+\alpha-\frac{1}{2}} + \frac{B_2}{\sqrt{s}-\frac{1}{2}} + \frac{B_3}{\sqrt{s}-\frac{3}{2}} + \frac{B_4}{(\sqrt{s}-\frac{3}{2})^2}, \tag{3.7}$$

   where B_1, B_2, B_3 and B_4 are constants.

3. if $\nu > \frac{1}{4}$, we have two complex roots.

$$r_1 = \beta_r + i\beta_i, \qquad r_2 = \beta_r - i\beta_i, \tag{3.8a,b}$$

   where β_r and β_i are given by

$$\beta_r = \frac{1-\nu}{2\nu}, \qquad \beta_i = \frac{\sqrt{4\nu-1}}{2\nu}. \tag{3.8c,d}$$

In this case $q(s)$ can be expressed as

$$q(s) = \frac{C_1}{\sqrt{s} + \alpha - \frac{1}{2}} + \frac{C_2}{\sqrt{s} - \frac{1}{2}} + \frac{C_3}{\sqrt{s} - r_1} + \frac{C_4}{\sqrt{s} - r_2}, \tag{3.9}$$

where C_1, C_2, C_3 and C_4 are constants.

In all three cases, constants A_j, B_j, and C_j $(j = 1, 2, 3, 4)$ can be computed by a standard method of partial fractions. We show these computations in Appendix A and omit here for brevity. In order to invert $q(s)$ to obtain the solution for Φ, we must be able to compute the inverse Laplace transform of terms of the form $\frac{1}{\sqrt{s}-\beta}$. It can be shown that if $\beta = \beta_r + i\beta_i$ is a complex number, then

$$\mathcal{L}^{-1}\{\frac{1}{\sqrt{s} - \beta}\} = \frac{1}{\sqrt{\pi\tau}} + \beta(I_1(\tau;\beta) + iI_2(\tau;\beta)), \tag{3.10a}$$

where I_1 and I_2 are defined by

$$\begin{aligned} I_1(\tau;\beta) &= e^{(\beta_r^2-\beta_i^2)\tau}\mathrm{erfc}(-\beta_r\sqrt{\tau})\cos(2\beta_r\beta_i\tau) \\ &\quad + 2\sqrt{\frac{\tau}{\pi}}\int_0^{\beta_i} e^{(s^2-\beta_i^2)\tau}\sin(\beta_i - s)ds, \end{aligned} \tag{3.10b}$$

$$\begin{aligned} I_2(\tau;\beta) &= e^{(\beta_r^2-\beta_i^2)\tau}\mathrm{erfc}(-\beta_r\sqrt{\tau})\sin(2\beta_r\beta_i\tau) \\ &\quad + 2\sqrt{\frac{\tau}{\pi}}\int_0^{\beta_i} e^{(s^2-\beta_i^2)\tau}\cos(\beta_i - s)ds. \end{aligned} \tag{3.10c}$$

The complementary error function appearing above is defined by

$$\mathrm{erfc}(\tau) = \frac{2}{\sqrt{\pi}}\int_\tau^\infty e^{-t^2}dt. \tag{3.10d}$$

Notice that in the case where β is a real number ($\beta_i = 0$), (3.10a) reduces to

$$\mathcal{L}^{-1}\{\frac{1}{\sqrt{s} - \beta}\} = \frac{1}{\sqrt{\pi\tau}} + \beta e^{\beta^2\tau}\mathrm{erfc}(-\beta\sqrt{\tau}). \tag{3.10e}$$

Finally, from (3.10e) one finds that for real number β,

$$\mathcal{L}^{-1}\{\frac{1}{(\sqrt{s} - \beta)^2}\} = \frac{2\beta\sqrt{\tau}}{\sqrt{\pi}} + (1 + 2\beta^2\tau)e^{\beta^2\tau}\mathrm{erfc}(-\beta\sqrt{\tau}). \tag{3.10f}$$

The derivations of (3.10a,b,c,f) are rather involved. They are detailed in Appendix B and omitted here for brevity.

From (3.10) we find

$$\mathcal{L}^{-1}\{\frac{1}{\sqrt{s}+\alpha-\frac{1}{2}}\} = \frac{1}{\sqrt{\pi\tau}} + (\frac{1}{2}-\alpha)\Big(I_1(\tau;\frac{1}{2}-\alpha) + iI_2(\tau;\frac{1}{2}-\alpha)\Big), \quad (3.11a)$$

$$\mathcal{L}^{-1}\{\frac{1}{\sqrt{s}-\frac{1}{2}}\} = \frac{1}{\sqrt{\pi\tau}} + \frac{1}{2}e^{\tau/4}\mathrm{erfc}(-\sqrt{\tau}/2). \quad (3.11b)$$

With use of (3.6), (3.7), (3.9) and (3.10), we now evaluate the inverse Laplace transform in (3.4) and obtain the following expressions for $\Phi(\tau)$:

1. if $\nu < \frac{1}{4}$,

$$\begin{aligned}\Phi(\tau) =&\Phi(0) + e^{-\tau/4}\Big\{A_1\mathcal{L}^{-1}\{\frac{1}{\sqrt{s}+\alpha-\frac{1}{2}}\} + A_2\mathcal{L}^{-1}\{\frac{1}{\sqrt{s}-\frac{1}{2}}\}\\ &+ A_3[\frac{1}{\sqrt{\pi\tau}} + r_1 e^{r_1^2\tau}\mathrm{erfc}(-r_1\sqrt{\tau})]\\ &+ A_4[\frac{1}{\sqrt{\pi\tau}} + r_2 e^{r_2^2\tau}\mathrm{erfc}(-r_2\sqrt{\tau})]\Big\}. \end{aligned} \quad (3.12)$$

2. if $\nu = \frac{1}{4}$,

$$\begin{aligned}\Phi(\tau) =&\Phi(0) + e^{-\tau/4}\Big\{B_1\mathcal{L}^{-1}\{\frac{1}{\sqrt{s}+\alpha-\frac{1}{2}}\} + B_2\mathcal{L}^{-1}\{\frac{1}{\sqrt{s}-\frac{1}{2}}\}\\ &+ B_3[\frac{1}{\sqrt{\pi\tau}} + \frac{3}{2}e^{9\tau/4}\mathrm{erfc}(-3\sqrt{\tau}/2)]\\ &+ B_4[\frac{3\sqrt{\tau}}{\sqrt{\pi}} + (1+9\tau/2)e^{9\tau/4}\mathrm{erfc}(-3\sqrt{\tau}/2)]\Big\}. \end{aligned} \quad (3.13)$$

3. if $\nu > \frac{1}{4}$,

$$\begin{aligned}\Phi(\tau) =&\Phi(0) + e^{-\tau/4}\Big\{C_1\mathcal{L}^{-1}\{\frac{1}{\sqrt{s}+\alpha-\frac{1}{2}}\} + C_2\mathcal{L}^{-1}\{\frac{1}{\sqrt{s}-\frac{1}{2}}\}\\ &+ 2C_r(\frac{1}{\sqrt{\pi\tau}} + \beta_r I_1 - \beta_i I_2) - 2C_i(\beta_i I_1 + \beta_r I_2)\Big\}, \end{aligned} \quad (3.14)$$

where C_r and C_i are the real and imaginary parts of C_3, respectively, and β_r, β_i, I_1, I_2, $\mathcal{L}^{-1}\{\frac{1}{\sqrt{s}+\alpha-\frac{1}{2}}\}$ and $\mathcal{L}^{-1}\{\frac{1}{\sqrt{s}-\frac{1}{2}}\}$ are given in (3.8c,d), (3.10b,c) and (3.11a,b).

The term $\frac{1}{\sqrt{\pi\tau}}$ in (3.12), (3.13) and (3.14) does not contribute singularity at $\tau = 0$ as one can show that these terms cancel out in all three cases listed above.

Stability Analysis:

We now discuss the stability property of the front perturbation, $\Phi(\tau)$. The following two limits for the complementary error function are useful,

$$\lim_{x\to\infty} \operatorname{erfc}(x) = 0, \qquad (3.15a) \qquad \lim_{x\to-\infty} \operatorname{erfc}(x) = 2. \qquad (3.15b)$$

1. For $\nu < \frac{1}{4}$, we study the solution, $\Phi(\tau)$ given in (3.12). We are interested in the four complementary error function terms associated with A_1, A_2, A_3, and A_4, respectively. First, from (3.11b), the term associated with A_2 is stable for all $\nu < \frac{1}{4}$. For the term associated with A_1, we study (3.11a) to show that both $I_1(\tau; \frac{1}{2} - \alpha)$ and $I_2(\tau; \frac{1}{2} - \alpha)$ are bounded. From (3.10b,c) we know that we need to focus on the amplitude, $e^{[(\frac{1}{2}-\alpha_r)^2-\alpha_i^2-\frac{1}{4}]\tau}\operatorname{erfc}\big((\alpha_r - \frac{1}{2})\sqrt{\tau}\big)$, of the sine and cosine terms since all other terms decay in time. If $(\frac{1}{2}-\alpha_r)^2-\alpha_i^2 \le \frac{1}{4}$ both exponential and complementary error functions are bounded. If $(\frac{1}{2}-\alpha_r)^2-\alpha_i^2 > \frac{1}{4}$, then $\alpha_r > 1$ and we use L'Hôpital's rule to obtain

$$\lim_{\tau\to\infty} \frac{\operatorname{erfc}\big((\alpha_r - \frac{1}{2})\sqrt{\tau}\big)}{e^{[\frac{1}{4}+\alpha_i^2-(\frac{1}{2}-\alpha_r)^2]\tau}} = \lim_{\tau\to\infty} \frac{(\frac{1}{2} - \alpha_r)e^{-(\frac{1}{2}-\alpha_r)^2\tau}/\sqrt{\pi\tau}}{[\frac{1}{4} + \alpha_i^2 - (\frac{1}{2} - \alpha_r)^2]e^{[\frac{1}{4}+\alpha_i^2-(\frac{1}{2}-\alpha_r)^2]\tau}} = 0. \quad (3.16)$$

   Therefore, the term associated with A_1 is bounded for all $\nu < \frac{1}{4}$. Finally, we study terms associated with A_3 and A_4. For $\nu < \frac{1}{4}$, we have $r_1 > r_2 > \frac{1}{2}$. The fact that both r_1 and r_2 are positive shows that the limits of the two complementary error functions $\operatorname{erfc}(-r_1\sqrt{\tau})$ and $\operatorname{erfc}(-r_2\sqrt{\tau})$ as $\tau \to \infty$ both equal 2. Since r_1 is greater than $\frac{1}{2}$ and $r_1 > r_2$, the exponential function $e^{(r_1^2-\frac{1}{4})\tau}$ is the dominant growing term as τ increases and cannot be cancelled by $e^{(r_2^2-\frac{1}{4})\tau}$. Therefore, if $\nu < \frac{1}{4}$, the front perturbation $\Phi(\tau)$ grows exponentially in time and the linearized problem is unstable for all initial conditions given in (2.15).

2. For $\nu = \frac{1}{4}$, a similar arguement as given above shows that terms associated with B_1 and B_2 in (3.13) are bounded and the dominant growing term comes

from the the term associated with B_4 and behaves like $\tau e^{2\tau}$. Therefore, the linearized problem is also unstable for all initial conditions for $\nu = \frac{1}{4}$.

3. For $\nu > \frac{1}{4}$, the same analysis shows that terms associated with C_1 and C_2 in (3.14) are bounded. For the terms associated with C_r and C_i, we study both $I_1(\tau;\beta)$ and $I_2(\tau;\beta)$ defined in (3.10b,c). As mentioned earlier, we need only to focus on the amplitude, $e^{(\beta_r^2-\beta_i^2-\frac{1}{4})\tau}\text{erfc}(-\beta_r\sqrt{\tau})$, of the sine and cosine terms, as all other terms decay in time. If $\beta_r^2 - \beta_i^2 - \frac{1}{4} \leq 0$, i.e., $\nu \geq \frac{1}{3}$, then the amplitude decays in time as the complementary error function is bounded. If $\beta_r^2 - \beta_i^2 - \frac{1}{4} > 0$ ($\nu < \frac{1}{3}$), then $\beta_r > 0$ so that $\lim_{\tau\to\infty} \text{erfc}(-\beta_r\sqrt{\tau}) = 2$, and the amplitude grows exponentially in time. Therefore, if $\nu < \frac{1}{3}$ the linearized problem is unstable for all initial conditions given in (2.15).

In summary, for the flat mode ($k = 0$) we have proved that for the initial condition given in (2.15), the linearized problem (2.4) is unstable for $\nu < \frac{1}{3}$ and stable for $\nu \geq \frac{1}{3}$.

3.2 <u>Non-flat mode case ($k \neq 0$):</u>

For the non-flat mode case, we must study (3.1) directly. First, we compute numerically the three roots of $p(s)$, where $p(s)$ is defined as a third-degree polynomial in $\sqrt{s}$ in (2.14). Denoting the three roots as r_1, r_2 and r_3, one finds that (3.1) becomes

$$\tilde{\Phi}(s - \frac{1}{4} - k^2) = -\frac{1}{\nu}\Big(\frac{-\nu s + (1-\nu)\sqrt{s} - \frac{1}{2}(1-\frac{\nu}{2})}{(\sqrt{s}-r_1)(\sqrt{s}-r_2)(\sqrt{s}-r_3)(\sqrt{s}+\frac{1}{2})}\Phi(0) - \frac{g_0}{(\sqrt{s}-r_1)(\sqrt{s}-r_2)(\sqrt{s}-r_3)(\sqrt{s}+\alpha-\frac{1}{2})}\Big). \tag{3.17}$$

Proceeding as in Section 3.1, a partial fraction expansion of (3.17) is carried out. Finally, solution of the front perturbation Φ is obtained by an inverse Laplace transform using formulas in (3.10).

From the stability analysis given in Section 3.1 we know that the unstable terms of the front perturbation Φ come from $I_1(\tau;\beta)$ and $I_2(\tau;\beta)$ defined in (3.10b,c) due to the complex root $r_1 = \beta_r + i\beta_i$ of $p(s)$ in (3.1). The amplitude of the sine and cosine terms is given by

$$e^{(\beta_r^2-\beta_i^2-\frac{1}{4}-k^2)\tau}\text{erfc}(-\beta_r\sqrt{\tau}). \tag{3.18}$$

One can show numerically that β_r is always positive in the parameter regime so that $\mathrm{erfc}(-\beta_r\sqrt{\tau})$ is bounded between 1 and 2. Therefore, the growth rate may be defined by

$$\text{Growth Rate} = \beta_r^2 - \beta_i^2 - \frac{1}{4} - k^2. \tag{3.19}$$

Letting the growth rate equal to zero, one obtains the neutral stability curves. We show these curves for different values of γ ($\gamma = 0, 0.05$ and 0.1) in Figure 2. In Figure 3 we plot the growth rate vs k for $\gamma = 0$ and $\nu = 0.35$, 0.335 and 0.32. As expected, the growth rate for $\nu = 0.35$ is always negative as it lies inside of the stable region. For $\nu = 0.335$, the growth rates are positive between $0.155 < k < 0.962$ and attain a maximum value of 0.1027 at $k = 0.595$. For $\nu = 0.32$, the growth rates are positive for $k < 1.255$ and attain a maximum value of 0.3203 at $k = 0.5876$. The behavior of growth rate for $\gamma \neq 0$ is similar to that in Figure 3 for $\gamma = 0$.

Notice that after computing the three roots of $p(s)$ numerically, one can obtain the following explicit expression of the behavior of a typical term in the reaction front perturbation,

$$e^{(\beta_r^2-\beta_i^2-\frac{1}{4}-k^2)\tau}\mathrm{erfc}(-\beta_r\sqrt{\tau})\left\{\begin{array}{c}\cos(2\beta_r\beta_i\tau)\\ \sin(2\beta_r\beta_i\tau)\end{array}\right\}. \tag{3.20}$$

In a linear stability analysis using Fourier theory, one obtains a typical term with exponential function multiplied by a sine or cosine function. Here in our study, the exponential term in (3.20) is multiplied not only by a sine or cosine function, but also by an additional complementary error function. As shown earlier, β_r is always positive and the complementary error function is bounded between 1 and 2 so that the stability property is not altered by this additional factor.

Finally, we study the solution of the temperature $W(\tau, \eta)$ defined in (2.8) and (2.5) to show that its behavior does not alter the linear stability property concluded from the study of the reaction front perturbation given above. First, if the reaction front perturbation Φ has a term that grows exponentially in time, then the linearized problem (2.4) is already unstable regardless of the behavior of the temperature W. On the other hand, if all the terms in the front solution Φ are decaying exponentially in time, we show that W is bounded as $\tau \to \infty$. This part of proof is rather involved. We detail it in Appendix C and omit here for brevity.

4. Concluding remarks

We have carried out a linear stability study for a simple solid combustion model. The linearized problem for the perturbations to a basic solution has been solved by the method of images. The explicit solution of the reaction front perturbation with the flat mode has been derived by the method of Laplace transforms. For the non-flat mode, we have computed roots of a third-degree polynomial numerically. Once the values of these roots are available, explicit expression of the solution for the reaction front perturbation can be obtained. A typical term in the front perturbation solution is shown to have an exponential function multiplied by a sine or a cosine function, as well as an additional complementary error function. Growth rates of such a typical term are studied. Neutral stability curves are shown in Figure 2, concuring with results obtained by earlier studies (e.g., Gross (1997)). Also, we have shown that the linear stability property of our model is determined by the solution of the reaction front perturbation and is not altered by the solution of the temperature distribution.

Finally, because our solution is obtained for arbitrary initial disturbances, it is now possible to study the transient behavior of the model.

REFERENCES

Carrier, G. F & Pearson, C. E. 1988 *Partial Differential Equations: Theory and Technique*, Academic Press, Inc.

Frankel, M. L. 1991 *IMA Volumes in Mathematics and Its Applications* **35**, 107–127.

Frankel, M. L. & Roytburd, V. 1994 *International Journal of Bifurcation and Chaos* **4**, 579–593.

Frankel, M. L., Roytburd, V. & Sivashinsky, G. I. 1994 *SIAM J. Appl. Math.* **54**, 1357–1374.

Gross, L. K. 1997 Weakly nonlinear analysis of interface propagation. Ph. D. thesis, Rensselaer Polytechnic Institute.

Kevorkian, J. 1990 *Partial Differential Equations: Analytical Solution Techniques*, 2nd edition, Springer-Verlag, 2000.

Matkowsky, B. J. & Sivashinsky, G. I. 1978 *SIAM J. Appl. Math.* **35**, 230–255.

Merzhanov, A. G. 1981 *Arch. Combustions* **1**, 23–48.

Volpert, Vit. A.,Volpert, Vl. A., Davtyan, S. P.,Megrabova, I. N. & Surkov, N. F. 1992 *SIAM J. Appl. Math.* **52**, 368–383.

Yu, J. & Kevorkian, J. 1992 *J. Fluid Mech.* **243**, 575–594.

Yu, J., Kevorkian, J. & Haberman, R. 1999 To appear in *Studies in Applied Mathematics*.

Appendix A

We compute constants A_j, B_j and C_j $(j = 1, 2, 3, 4)$ in (3.6), (3.7) and (3.8) by a standard method of partial fraction. The results are listed as follows:

$$A_1 = -\frac{1}{\alpha(\alpha + (\alpha - 1)^2\nu)}, \qquad (A.1a), \qquad A_2 = \frac{1}{\alpha\nu}, \qquad (A.1b)$$

$$A_3 = \frac{4\nu^2}{\sqrt{1-4\nu}(1 - 2\nu + \sqrt{1-4\nu})(1 - 2\nu + \sqrt{1-4\nu} + 2\alpha\nu)}, \qquad (A.1c)$$

$$A_4 = -\frac{4\nu^2}{\sqrt{1-4\nu}(1 - 2\nu - \sqrt{1-4\nu})(1 - 2\nu - \sqrt{1-4\nu} + 2\alpha\nu)}, \qquad (A.1d)$$

$$B_1 = -\frac{4}{\alpha(\alpha+1)^2}, \qquad (A.2a), \qquad B_2 = \frac{4}{\alpha}, \qquad (A.2b)$$

$$B_3 = -\frac{4(2+\alpha)}{(\alpha+1)^2}, \qquad (A.2c), \qquad B_4 = \frac{4}{1+\alpha}, \qquad (A.2d)$$

$$C_1 = -\frac{1}{\alpha(\alpha + (\alpha - 1)^2\nu)}, \qquad (A.3a), \qquad C_2 = \frac{1}{\alpha\nu}, \qquad (A.3b)$$

$$C_3 = \frac{4\nu^2 i}{\sqrt{4\nu - 1}(1 - 2\nu + i\sqrt{4\nu - 1})(1 - 2\nu + 2\alpha\nu + i\sqrt{4\nu - 1})}, \qquad (A.3c)$$

$$C_4 = -\frac{4\nu^2 i}{\sqrt{4\nu - 1}(1 - 2\nu - i\sqrt{4\nu - 1})(1 - 2\nu + 2\alpha\nu - i\sqrt{4\nu - 1})}. \qquad (A.3d)$$

Appendix B

In order to compute the inverse Laplace transform given in (3.10), we notice that

$$\mathcal{L}\{\frac{x}{\sqrt{4\pi}\tau^{3/2}}e^{-\frac{x^2}{4\tau}}\} = e^{-x\sqrt{s}}, \qquad (B.1)$$

as given, for example, in Section 2 of Carrier & Pearson (1988). Multiplying both sides of (B.1) by $e^{\beta x}$ and integrating with respect to x from 0 to ∞, one finds that

$$\mathcal{L}\{\int_0^\infty e^{\beta x}\frac{x}{\sqrt{4\pi}\tau^{3/2}}e^{-\frac{x^2}{4\tau}}dx\} = \frac{1}{\sqrt{s}-\beta}, \qquad (B.2)$$

where $\beta = \beta_r + i\beta_i$ is a complex number.

We now evaluate the integral on the left hand side of (B.2),

$$\int_0^\infty e^{\beta x}\frac{x}{\sqrt{4\pi}\tau^{3/2}}e^{-\frac{x^2}{4\tau}}dx = \int_0^\infty e^{\beta_r x}\frac{x}{\sqrt{4\pi}\tau^{3/2}}e^{-\frac{x^2}{4\tau}}(\cos\beta_i x + i\sin\beta_i x)dx. \quad (B.3)$$

Define

$$I_3 = \int_0^\infty e^{\beta_r x - \frac{x^2}{4\tau}}\cos\beta_i x dx, \quad (B.4a), \quad I_4 = \int_0^\infty e^{\beta_r x - \frac{x^2}{4\tau}}\sin\beta_i x dx. \qquad (B.4b)$$

$$I_5 = \int_0^\infty \frac{x\cos\beta_i x}{2\tau}e^{\beta_r x - \frac{x^2}{4\tau}}dx, \quad (B.4c), \quad I_6 = \int_0^\infty \frac{x\sin\beta_i x}{2\tau}e^{\beta_r x - \frac{x^2}{4\tau}}dx. \quad (B.4d)$$

Differentiating (B.4a,b) with respect to β_i, one finds that

$$\frac{dI_3}{d\beta_i} = -2\tau I_6, \qquad (B.5a), \qquad \frac{dI_4}{d\beta_i} = 2\tau I_5. \qquad (B.5b)$$

On the other hand, integration by parts in (B.4c,d) shows that

$$I_5 = 1 - \beta_i I_4 + \beta_r I_3, \qquad (B.6a), \qquad I_6 = \beta_i I_3 + \beta_r I_4. \qquad (B.6b)$$

Combining (B.5) with (B.6) gives the following system of ordinary differential equations for I_3 and I_4 as functions of β_i,

$$\frac{dI_3}{d\beta_i} = -2\tau\beta_i I_3 - 2\tau\beta_r I_4, \qquad (B.7a)$$

$$\frac{dI_4}{d\beta_i} = 2\tau\beta_r I_3 - 2\tau\beta_i I_4 + 2\tau. \qquad (B.7b)$$

The general solution of (B.7) is given by

$$\begin{pmatrix} I_3(\beta_i) \\ I_4(\beta_i) \end{pmatrix} = \Phi(\beta_i)\Big(\mathbf{C} + \int^{\beta_i} \Phi^{-1}(s) \begin{pmatrix} 0 \\ 2\tau \end{pmatrix} ds\Big), \qquad (B.8)$$

where $\Phi(\beta_i)$ is the fundamental matrix solution of the homogenous equation given by

$$\Phi(\beta_i) = e^{-\tau\beta_i^2} \begin{pmatrix} \cos(2\tau\beta_r\beta_i) & -\sin(2\tau\beta_r\beta_i) \\ \sin(2\tau\beta_r\beta_i) & \cos(2\tau\beta_r\beta_i) \end{pmatrix}, \qquad (B.9)$$

and $\Phi^{-1}(\beta_i)$ the inverse of $\Phi(\beta_i)$, i.e.,

$$\Phi^{-1}(\beta_i) = e^{\tau\beta_i^2} \begin{pmatrix} \cos(2\tau\beta_r\beta_i) & \sin(2\tau\beta_r\beta_i) \\ -\sin(2\tau\beta_r\beta_i) & \cos(2\tau\beta_r\beta_i) \end{pmatrix}. \qquad (B.10)$$

To completely determine I_3 and I_4 in (B.8) we evaluate I_3 and I_4 at $\beta_i = 0$ and find that

$$I_3(0) = e^{\tau\alpha^2}\sqrt{\pi\tau}\mathrm{erfc}(-\sqrt{\tau}\alpha), \qquad (B.11a); \qquad I_4(0) = 0. \qquad (B.11b)$$

Substituting (B.9), (B.10) and (B.11) into (B.8), one finds that

$$\begin{aligned} I_3 &= e^{(\beta_r^2-\beta_i^2)\tau}\sqrt{\pi\tau}\mathrm{erfc}(-\sqrt{\tau}\alpha)\cos(2\tau\beta_r\beta_i) \\ &\quad + 2\tau\int_0^{\beta_i} e^{-\tau(\beta_i^2-s^2)}\sin(\beta_i - s)ds, \qquad (B.12a) \\ I_4 &= e^{(\beta_r^2-\beta_i^2)\tau}\sqrt{\pi\tau}\mathrm{erfc}(-\sqrt{\tau}\alpha)\sin(2\tau\beta_r\beta_i) \\ &\quad + 2\tau\int_0^{\beta_i} e^{-\tau(\beta_i^2-s^2)}\cos(\beta_i - s)ds. \qquad (B.12b) \end{aligned}$$

From (B.6), we find that

$$I_5 + iI_6 = 1 + \beta(I_3 + iI_4). \qquad (B.13)$$

Now, dividing both sides of (B.13) by $\sqrt{\pi\tau}$ and using (B.3) and (B.4), we obtain (3.10a,b,c).

To derive (3.10f), we use the following two properties of Laplace transforms:

1.) if $\mathcal{L}\{f(\tau)\} = F(s)$, then $\mathcal{L}\{f'(\tau)\} = sF(s) - f(0)$, 2.) if $\mathcal{L}\{f(\tau)\} = F(s)$, then $\mathcal{L}\{-\tau f(\tau)\} = \frac{dF}{ds}$.

Denote the right hand side of (3.10e) as $f(\tau)$, i.e.,

$$f(\tau) = \frac{1}{\sqrt{\pi\tau}} + \beta e^{\beta^2\tau}\operatorname{erfc}(-\beta\sqrt{\tau}). \tag{B.14}$$

Using property 2, one finds that (3.10e) becomes $\mathcal{L}\{f(\tau)\} = \frac{1}{\sqrt{s}-\beta}$ and

$$\mathcal{L}\{-\tau f(\tau)\} = -\frac{1}{2\sqrt{s}}\frac{1}{(\sqrt{s}-\beta)^2}. \tag{B.15}$$

Notice that $-\tau f(\tau)|_{\tau=0} = 0$, so that property 1 shows that

$$\mathcal{L}\left\{\frac{d[-\tau f(\tau)]}{d\tau}\right\} = s\mathcal{L}\{-\tau f(\tau)\} - 0. \tag{B.16}$$

Using (B.15), one can compute the right hand side of (B.16) to obtain

$$sL\{-\tau f(\tau)\} = -\frac{1}{2}\frac{\sqrt{s}}{(\sqrt{s}-\beta)^2} = -\frac{1}{2}\frac{1}{\sqrt{s}-\beta} - \frac{\beta}{2}\frac{1}{(\sqrt{s}-\beta)^2}. \tag{B.17}$$

Carrying out the derivative on the left hand side of (B.16), we find that

$$\mathcal{L}\left\{\frac{d[-\tau f(\tau)]}{d\tau}\right\} = -\mathcal{L}\{f(\tau)\} - \mathcal{L}\{-\tau f'(\tau)\}. \tag{B.18}$$

Substituting (B.17) and (B.18) into (B.16), taking the inverse Laplace transform on both sides and solving for $\mathcal{L}^{-1}\{\frac{1}{(\sqrt{s}-\beta)^2}\}$, one finds that

$$\mathcal{L}^{-1}\{\frac{1}{(\sqrt{s}-\beta)^2}\} = \frac{1}{\beta}f(\tau) + \frac{2}{\beta}\tau f'(\tau). \tag{B.19}$$

Using (B.14) and calculating the right hand side of (B.19) we obtain (3.10f).

Appendix C

We show here that if all the terms in the reaction front solution Φ decay exponentially in time, the temperature W is bounded as $\tau \to \infty$. From (2.5), we find that $W = e^{-(k^2+\frac{1}{4})\tau}u$, where u is defined in (2.8). There are two terms in (2.8) and we study the second term first. With the initial condition $g^*(\eta)$ defined in (2.15), the second integral in (2.8) will contribute a term in W that behaves as

$$\frac{e^{-(k^2+\frac{1}{4})\tau}}{\sqrt{4\pi\tau}}\int_0^\infty [e^{-(\eta-x)^2/4\tau} - e^{-(\eta+x)^2/4\tau}]e^{\tilde{\alpha}x}dx, \qquad (C.1)$$

where $\tilde{\alpha}$ is $\frac{1}{2}$ or $\frac{1}{2}-\alpha$.

If $\tilde{\alpha} = \frac{1}{2}$, a change of variable $v = \frac{x-\eta-\tau}{2\sqrt{\tau}}$ shows that the first term of (C.1) becomes

$$\frac{e^{\frac{\eta}{2}-k^2\tau}}{2}\mathrm{erfc}(-\frac{\tau+\eta}{2\sqrt{\tau}}), \qquad (C.2a)$$

and $v = \frac{x+\eta-\tau}{2\sqrt{\tau}}$ in the second term of (C.1) gives

$$\frac{e^{-\frac{\eta}{2}-k^2\tau}}{2}\mathrm{erfc}(-\frac{\tau-\eta}{2\sqrt{\tau}}). \qquad (C.2b)$$

Using the limit in (3.15b) one finds that both terms in (C.2a, b) are bounded as $\tau \to \infty$.

If $\tilde{\alpha} = \frac{1}{2} - \alpha$ with $\alpha = \alpha_r + i\alpha_i$, terms in (C.1) are bounded by

$$\frac{e^{-(k^2+\frac{1}{4})\tau}}{\sqrt{4\pi\tau}}\int_0^\infty [e^{-(\eta-x)^2/4\tau} - e^{-(\eta+x)^2/4\tau}]e^{(\frac{1}{2}-\alpha_r)x}dx, \qquad (C.3)$$

since $|e^{-i\alpha_i x}| \leq 1$. A change of variable $v = \frac{x-\eta-(1-2\alpha_r)\tau}{2\sqrt{\tau}}$ shows that the first term of (C.3) becomes

$$\frac{e^{(\frac{1}{2}-\alpha_r)\eta+(\alpha_r^2-\alpha_r-k^2)\tau}}{2}\mathrm{erfc}\Big(-\frac{\eta}{2\sqrt{\tau}} - (\frac{1}{2}-\alpha_r)\sqrt{\tau}\Big), \qquad (C.4a)$$

and $v = \frac{x+\eta-(1-2\alpha_r)\tau}{2\sqrt{\tau}}$ in the second term of (C.3) gives

$$\frac{e^{-(\frac{1}{2}-\alpha_r)\eta+(\alpha_r^2-\alpha_r-k^2)\tau}}{2}\mathrm{erfc}\Big(\frac{\eta}{2\sqrt{\tau}} - (\frac{1}{2}-\alpha_r)\sqrt{\tau}\Big). \qquad (C.4b)$$

If $\alpha_r < 1$, then both terms in (C.4a, b) tend to zero as $\tau \to \infty$, due to the $e^{(\alpha_r^2-\alpha_r-k^2)\tau}$ term. If $\alpha_r \geq 1$, then a L'Hôpital's rule similar to that in (3.16) shows that both terms in (C.4a, b) also tend to zero as $\tau \to \infty$. Therefore, all terms contributed to W from the second term of (2.8) are bounded as $\tau \to \infty$.

We now study terms contributed to W from the first term in (2.8). Due to the boundness of the sine, cosine and complementary error functions, a typical term of Φ, Φ' and Φ'' is bounded by $e^{\tilde{\beta}\tau}$ with $\tilde{\beta} < 0$ (assuming all the terms in Φ decay exponentially in time). Using the definition of H in (2.6), one finds a typical term for W from the first term of (2.8) as

$$\int_0^\tau e^{\tilde{\beta}t} \frac{e^{-(k^2+\frac{1}{4})\tilde{t}}}{\sqrt{4\pi\tilde{t}}} \int_0^\infty [e^{-(\eta-x)^2/4\tilde{t}} - e^{-(\eta+x)^2/4\tilde{t}}] e^{\pm\frac{x}{2}} dx dt, \qquad (C.5)$$

where $\tilde{t} = \tau - t$. Similar to the study of (C.1), we make change of variable, $v = \frac{x \pm \eta - \tilde{t}}{2\sqrt{\tilde{t}}}$ in the inner integral with respect to x so that (C.5) becomes

$$\int_0^\tau \frac{e^{\tilde{\beta}t \pm \frac{\eta}{2} - k^2(\tau-t)}}{2} \mathrm{erfc}(-\frac{\tau - t \pm \eta}{2\sqrt{\tau - t}}) dt. \qquad (C.6)$$

Integral (C.6) is bounded by

$$e^{\pm\frac{\eta}{2}} \int_0^\tau e^{\tilde{\beta}t} dt. \qquad (C.7)$$

With $\tilde{\beta} < 0$, term (C.7) is bounded as $\tau \to \infty$. Therefore, all terms contributed to W from the first term of (2.8) are bounded as $\tau \to \infty$.