Weakly Nonlinear Dynamics of Interface Propagation

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Abstract

A simple conceptual description of condensed-phase combustion, explosive solidification, and certain other exothermic phenomena can be given via a free boundary model with a nonlinear kinetic condition at the free boundary. For a wide range of parametric regimes, the reaction front exhibits a great variety of spatial patterns and instabilities. In [1], we did a linear stability analysis of interfaces that move along a two-dimensional semi-infinite strip with thermally insulated edges. Here we use the normal-mode method to perform a weakly nonlinear analysis for the development of transverse instabilities in the strip. The asymptotic analysis leads to the derivation of ordinary differential equations of Landau-Stuart type for the slowly-varying amplitudes of linearly unstable modes. We focus on a strip in which two eigenmodes lose stability at the same value of a parameter related to the activation energy. Such a case gives rise to nontrivial couplings between the amplitude equations, and the two unstable modes compete for dominance. Based on the bifurcation analysis of the amplitude equations, we classify the front configurations that will emerge for any given choice of the kinetics parameter.

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1 Introduction

In this article, we do a weakly nonlinear analysis to predict the salient features of the self-propagation of an interface. The underlying free-boundary model involves the heat equation on a semi-infinite strip with insulated edges and a nonlinear kinetic condition imposed on the moving boundary. Here we focus on the dynamics in the case of a strip that supports two kinds of traveling fronts: flat and wavy. We partition the values of the kinetics parameter into intervals that correspond to eventual dominance of a single mode versus the persistent presence of both.

One physical setting we consider is self-propagating high-temperature synthesis (SHS), an application of solid combustion. SHS is an efficient means of synthesizing high-quality ceramic materials and metallic alloys. (See [2, 3, 4].) The technique is to send a flame wave through powdered ingredients, converting them into the desired product. Our model also describes explosive solidification, in which thin layers of amorphous film undergo rapid crystallization, initiated by a laser pulse or stylus impact. (See [5].)

In both contexts, the thermal feedback between the chemical kinetics and the heat diffusion results in the sustainability of a traveling wave. At the same time, different parametric regimes lead to a wide variety of oscillatory behaviors, some leading to chaos. Merzhanov *et al.* [6] discovered such phenomena experimentally, and numerical studies have been done in, for example, [7, 8, 9, 10].

Many studies have systematically distinguished among the stable and various unstable behaviors. Aldushin and Kasparyan [11] contributed substantially to the understanding of stability in the broad context of combustion in 1979. Their bifurcation diagram identified regions of thermal instability, as well as of diffusion and fluid-dynamic instability, in the plane of Lewis number versus Zeldovich number.

More recent stability studies in combustion include [12, 13, 14, 15, 16], which take advantage of large activation energy asymptotics. Stability of combustion throughout a solid cylindrical sample is analyzed in [17]. Intricate bifurcation analyses [18, 19] have also classified the interactions of clockwise and counterclockwise spinning waves on the surface of a cylinder. Margolis' review paper [18] includes a thorough discussion of resonance phenomena, treating sample radii that yield close, as well as equal, eigenvalues. We note that the present stability study concerns a strip, rather than a cylinder. Studying a variety of geometries is valuable because combustion synthesis can be used to create such diverse products (*e.g.* abrasives, cutting tools, shape-memory alloys, ceramic engine parts, and nuclear safety shields [20]).

The strong dependence of the reaction rate on the temperature is crucial to modeling the exothermic phenomena under consideration. The bulk of the chemical reaction and heat release occur in a narrow zone. In [21], Matkowsky and Sivashinsky derived a free-boundary description of condensed-phase combustion as an asymptotic limit of the reaction-diffusion model. Margolis extended the approach in [22] to include gasless combustion processes in which the reactant melts prior to reacting. Melting is included in the stability analyses [18, 19].

In contrast to the articles [18, 19], this paper uses a one-sided version of the model in [21]. As in [23], the region ahead of the front is taken as the problem domain. In rapid solidification, such a model has been suggested in many different contexts, for example to describe an impurity controlled crystallization. (See [24] and references cited therein.) A very similar model for laser-induced evaporation from the surface of metals is used in [25]. In solid combustion, one can interpret a one-sided model as representing a reaction whose product has a very low heat conductivity. Comparisons between the one-sided and two-sided models, *e.g.* [26, 27], demonstrate that their dynamics are virtually identical, suggesting that the heat transfer behind the flame front (in the burned matter) and behind the amorphous-crystalline interface (in the crystalline phase) are qualitatively unimportant.

In Section 2 we give the governing equations. In Section 3 we do a routine linear stability analysis (*cf.* [28, 18, 19, 29]). Because we consider no heat flux at strip edges, normal-mode solutions for perturbations of the temperature and interface position involve cosine factors. They have the form

$$w = e^{\lambda\tau}\cos(k_j\xi)g(\eta;\lambda,\nu), \quad \phi = e^{\lambda\tau}\cos(k_j\xi), \tag{1.1}$$

where ξ is the transverse direction on a strip of width L, η is the longitudinal coordinate in a front-attached system, and $k_j = j\pi/L$, $j = 0, 1, 2, \ldots$. We also define a neutral stability curve for the material parameter ν, which is inversely proportional to the activation energy. The linear stability analysis of Section 3 appears in [30] and is reproduced here for completeness. (Different approaches to linear stability analysis are given in [1] and [31], in which the onset of linear instabilities can be examined from rather wide classes of initial conditions.) In Section 4 we derive and solve the adjoint linear problem.

The weakly nonlinear analysis occurs in Section 5. We focus first on the combustion case in which the strip of material has width π, resulting in the modes $k_0 = 0$ and $k_1 = 1$ losing stability at the same critical value $\nu_c = 1/3$. We examine the competition of the two slightly unstable flat and wavy modes.

Our method follows the classical Landau-Stuart theory of hydrodynamic stability, which dates back as far as the 1940's [32, 33, 34]. We note that the approach has been used in many different contexts. One example is reacting shock waves, for which the model involves a complex free-boundary problem for a system of nonlinear hyperbolic equations with source terms [35]. The technique was also used in [18, 19] for the two-sided solid combustion model with periodic boundary conditions. The idea is to study the evolution of the linearized eigenmodes modulated by slowly-varying, complex-valued amplitude functions $A_j(\epsilon\tau, \epsilon^2\tau)$, $j = 0, 1$. That is, we investigate temperature and interface position perturbed about a basic traveling-wave solution. Namely,

$$\begin{aligned} u(\xi,\eta,\tau,\epsilon\tau,\epsilon^2\tau) &= e^{-\eta} + \epsilon\sum_{j=0}^{1} A_j(\epsilon\tau,\epsilon^2\tau)e^{i\omega_j\tau}\cos(k_j\xi)g(\eta;i\omega_j,\nu_c) \\ &+ \epsilon^2 w_2(\xi,\eta,\tau,\epsilon\tau,\epsilon^2\tau) + \cdots + \mathrm{CC}, \\ f(\xi,\tau,\epsilon\tau,\epsilon^2\tau) &= \tau + \epsilon\left\{\sum_{j=0}^{1}[A_j(\epsilon\tau,\epsilon^2\tau)e^{i\omega_j\tau}\cos(k_j\xi)] + \frac{1}{2}B(\epsilon\tau,\epsilon^2\tau)\right\} \\ &+ \epsilon^2\phi_2(\xi,\tau,\epsilon\tau,\epsilon^2\tau) + \cdots + \mathrm{CC}, \end{aligned} \tag{1.2}$$

where $i\omega_j$ is a purely imaginary discrete eigenvalue λ, $g(\eta; i\omega_j, \nu_c)$ is the the corresponding eigenfunction, "CC" stands for complex-conjugate terms, and $\epsilon^2 = \nu_c - \nu = 1/3 - \nu$ is small. $B(\epsilon\tau, \epsilon^2\tau)$ is a (real-valued) amplitude function modulating the normal-mode solution $\phi = 1$.

In this paper, the subsequent strategy will consist of deriving constraints on the slowly-varying amplitudes A_j and B, which follow from inserting the ansatz (1.2) for the solution into the nonlinear equation and making systematic use of the method of multiple scales. (See, for example, Kevorkian and Cole [36].) The principal idea of the method is to consider the time variables τ, $\epsilon\tau$, and $\epsilon^2\tau$ as independent variables. Consecutive terms of the perturbation series satisfy linear inhomogeneous equations of the form

$$\partial u^j/\partial\tau + \mathcal{L}(u^j, \phi^j) = M_j(u^{j-1}, \phi^{j-1}, u^{j-2}, \phi^{j-2}, \ldots), \tag{1.3}$$

where the left-hand side is identical to the linearized operator in the eigenmode equation, while the right-hand side contains appropriate nonlinear terms (namely those that enter with a factor of ϵ^j) plus terms with appropriate derivatives of amplitudes with respect to slow times. According to Fredholm's alternative, the equation in (1.3) has a nonsecular (bounded in time) solution if the right-hand side is orthogonal to the corresponding eigenfunction of the adjoint linear operator. The orthogonality condition requires integration in ξ, η, and τ, which leads to a relation among the amplitudes. It should be noted that for the free-boundary problem at hand, the realization of the scheme just sketched is far from trivial, in particular in treating boundary conditions.

In the order ϵ^3 of the perturbation series, we obtain the system of Landau-Stuart equations

$$\frac{dA_j}{dt_2} = \chi_j A_j + \beta_{j,1} A_j^2 \bar{A}_j + \beta_{j,2} A_j A_{1-j} \bar{A}_{1-j}, \quad j = 0, 1 \tag{1.4}$$

from the solvability condition, where $t_2 = \epsilon^2\tau$ is the "slow" time, χ_j is determined by the derivative of the eigenvalue with respect to ν, and the Landau coefficients $\beta_{j,n}$ are functions of a kinetics parameter. The ordinary differential equations in (1.4) completely determine the dynamics of the unstable modes $A_j(t_2)e^{i\omega_j t_0}\cos k_j\xi$, $j = 0, 1$, subject to both nonlinear interaction and self-interaction. One can reduce the system (1.4) to a real system for the magnitudes $|A_0|^2$, $|A_1|^2$, in which the dynamics of the competition between the flat and wavy modes depend on the relationships among the coefficients of the system and are determined by the kinetic function. In Section 6 we investigate the qualitative weakly nonlinear behavior of the system for all values of a kinetics parameter σ.

2 Governing equations

We seek the temperature $u(x, y, t)$ and the interface position $\Gamma(t) = \{(x, y) \,|\, y = f(x, t)\}$ that satisfy the appropriately non-dimensionalized free boundary problem

$$u_t = \nabla^2 u, \quad 0 < x < L, \quad y > f(x, t), \quad t > 0, \tag{2.1}$$

$$u|_\Gamma = G(V) - \gamma\kappa, \quad t > 0, \tag{2.2}$$

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\Gamma} = -V, \quad t > 0. \tag{2.3}$$

As for the boundary conditions at the edges of the strip, we require the zero heat flux and zero contact angle conditions

$$u_x(0, y, t) = u_x(L, y, t) = 0, \quad \text{and} \quad f_x(0, t) = f_x(L, t) = 0. \tag{2.4}$$

Here $\mathbf{n}$ is the unit normal to the interface, V is the normal velocity of the interface, and κ is the interface curvature, *i.e.*

$$\mathbf{n} = \frac{(-f_x, 1)}{(1 + f_x^2)^{1/2}}, \quad V = (0, f_t) \cdot \mathbf{n} = \frac{f_t}{(1 + f_x^2)^{1/2}}, \quad \kappa = \frac{f_{xx}}{(1 + f_x^2)^{3/2}}.$$

The surface tension term $\gamma\kappa$ (the Gibbs-Thomson effect) in (2.2) appears only in the solidification problems, where the interface separates the liquid and the solid forms of the substance. In the combustion context $\gamma = 0$.

In addition, the temperature satisfies the condition

$$u \to 0 \text{ as } y \to \infty; \tag{2.5}$$

the ambient temperature is normalized to zero at infinity. We also note that the conservation of energy condition in (2.3) can be rewritten in the "standard free interface" form as

$$f_x u_x|_{\Gamma} - u_y|_{\Gamma} = f_t. \tag{2.6}$$

The second relation in (2.4) requires some explanation. It is not clear in advance which extra condition on the interface at the edges of the strip is necessary. It can be shown that for the two-sided model on the domain $-\infty < y < \infty$ the condition in (2.4) is the only one consistent with the other boundary conditions. Thus, it appears to be a natural condition to be retained for the one-sided model.

It is convenient to rewrite the non-equilibrium interface condition (2.2) in the form

$$u|_{\Gamma} = 1 + \nu K(V) - \gamma\kappa. \tag{2.7}$$

We will assume that the function $K(V) = (G(V) - 1)/\nu$ is normalized in such a way that $K(1) = 0$, $K'(1) = 1$, which can be achieved by rescaling the variables. In the context of combustion the parameter ν is inversely proportional to the activation energy of the exothermic chemical reaction that occurs at the interface.

Of principal interest for the SHS modeling is the Arrhenius kinetics. (See [3, 37].) Then, with appropriate non-dimensionalization, the velocity of propagation is related to the interface temperature by

$$V = \exp\left[\left(\frac{1}{\nu}\right) \frac{(u-1)}{\sigma + (1-\sigma)u}\right],$$

where $0 < \sigma < 1$ is the ambient temperature non-dimensionalized by the adiabatic temperature of combustion products. Correspondingly, the kinetic function in the boundary condition (2.7) takes the form

$$K(V) = \frac{\ln(V)}{1 - (1-\sigma)\nu \ln(V)}. \tag{2.8}$$

In the sequel, we will partition the continuum of ν values into stable and unstable regimes. We will also show the role that σ plays in determining the dynamics in the weakly nonlinear setting.

3 Linear stability analysis

For the stability analysis we reformulate the problem in the front-attached coordinate frame

$$\xi = x, \quad \eta = y - f(x,t), \quad \tau = t.$$

The problem (2.1), (2.6), (2.7) then takes the form

$$u_t = u_{\xi\xi} + (1 + f_\xi^2)u_{\eta\eta} - 2f_\xi u_{\xi\eta} + (f_t - f_{\xi\xi})u_\eta, \; 0 < \xi < L, \; \eta > 0, \; t > 0, \tag{3.1}$$

$$u|_\Gamma = u(\xi, 0, t) = 1 + \nu K(V) - \gamma\kappa, \tag{3.2}$$

$$u_\eta(\xi, 0, t) = \frac{-f_t(\xi, t) + f_\xi(\xi, t)u_\xi(\xi, 0, t)}{1 + f_\xi^2(\xi, t)}. \tag{3.3}$$

The zero heat flux and zero contact angle conditions at the edges of the strip and the zero condition at infinity from Section 2 are

$$u_\xi|_{\xi=0,L} = f_\xi|_{\xi=0,L} = \lim_{\eta\to\infty} u = 0. \tag{3.4}$$

The free boundary problem (3.1)–(3.4) admits a traveling-wave solution

$$w_0(\xi, \eta, \tau) = \exp(-\eta), \quad f_0(\xi, \tau) = \tau.$$

It is a flat front propagating with velocity 1 and is the only traveling-wave solution for $K(V)$ monotone.

The linearization about the steady solution follows easily from the substitution of $u = w_0 + \epsilon w$, $f = f_0 + \epsilon\phi$ into the governing equations. It has the form

$$\frac{\partial w}{\partial \tau} + \mathcal{L}(w, \phi) = 0 \tag{3.5}$$

with the boundary conditions at the front

$$\mathcal{M}(w, \phi) = 0, \quad \mathcal{N}(w, \phi) = 0 \tag{3.6}$$

and at the walls

$$\left.\frac{\partial \phi}{\partial \xi}\right|_{\xi=0,L} = 0, \quad \left.\frac{\partial w}{\partial \xi}\right|_{\xi=0,L} = 0 \tag{3.7}$$

and at the cold edge

$$\lim_{\eta\to\infty} w = 0, \tag{3.8}$$

where

$$\mathcal{L}(w, \phi) = -\frac{\partial^2 w}{\partial \xi^2} - \frac{\partial^2 w}{\partial \eta^2} - \frac{\partial w}{\partial \eta} + e^{-\eta}\left(\frac{\partial \phi}{\partial \tau} - \frac{\partial^2 \phi}{\partial \xi^2}\right), \tag{3.9}$$

$$\mathcal{M}(w,\phi) = w|_{\eta=0} - \nu\frac{\partial\phi}{\partial\tau} + \gamma\frac{\partial^2\phi}{\partial\xi^2}, \quad \text{and} \quad \mathcal{N}(w,\phi) = \left.\frac{\partial w}{\partial\eta}\right|_{\eta=0} + \frac{\partial\phi}{\partial\tau}. \tag{3.10}$$

A separated-variables solution of the linearized problem above is the normal-mode solution in (1.1), where $k_j = j\pi/L$ for the strip width L, and $j = 0, 1, 2, \ldots$. Also λ and $g(\eta;\lambda,\nu)$ satisfy the eigenvalue problem

$$g'' + g' - (k_j^2 + \lambda)g = (k_j^2 + \lambda)e^{-\eta}, \tag{3.11}$$

$$g|_{\eta=0} = \nu\lambda + \gamma k_j^2, \tag{3.12}$$

$$g'|_{\eta=0} = -\lambda, \tag{3.13}$$

$$\lim_{\eta\to\infty} g(\eta;\lambda,\nu) = 0. \tag{3.14}$$

The essential spectrum of (3.11)–(3.14) fills the parabola $\Re\lambda = -(\Im\lambda)^2 - k_j^2$. All of its elements satisfy $\Re\lambda < 0$, so the basic solution $(w_0,\ f_0)$ is linearly stable for such eigenvalues.

The discrete spectrum satisfies the equation

$$(\lambda + k_j^2)(\lambda\nu + \gamma k_j^2 + 1)^2 + \left(\lambda(\nu - 1) + \gamma k_j^2\right)(\lambda + 1) = 0. \tag{3.15}$$

The dispersion relation (3.15) has one nonpositive real root and two complex-conjugate roots λ. To emphasize the dependence of λ on k_j, we now write it as λ_j. The corresponding eigenfunctions are

$$g(\eta;\lambda_j,\nu) = (1 + \nu\lambda_j + \gamma k_j^2)\exp\left(-\left(1 + \sqrt{1 + 4(k_j^2 + \lambda_j)}\right)\frac{\eta}{2}\right) - \exp(-\eta). \tag{3.16}$$

The basic solution loses linear stability when the real part of the complex-conjugate eigenvalues increases through the value zero.

By setting $\Re\lambda = 0$ in the dispersion relation (3.15) and eliminating $\Im\lambda$, we define the critical value ν_c of ν as an implicit function of k_j and γ:

$$\begin{aligned}(2\gamma k_j^2\nu_c + 3\nu_c + \nu_c^2 k_j^2 - 1)\left(k_j^4(2\nu_c\gamma + \gamma^2) + k_j^2(3\gamma + 2\nu_c) + \nu_c\right) \\ - \quad \nu_c^2 k_j^2\left(\gamma^2 k_j^4 + 2\gamma k_j^2 + 1 + \gamma\right) = 0.\end{aligned} \tag{3.17}$$

Replacing k_j by a continuum of values k, we plot ν_c against k for various values of γ to produce the neutral stability curves of Figure 1. Note that, as in many other nonlinear problems of practical interest, the curves are bell-shaped. The region of stability corresponding to each neutral stability curve lies above the curve; the instability region lies below it. That is, for fixed values of k and γ, the corresponding pair of complex-conjugate eigenvalues crosses the imaginary axis in the λ plane as ν decreases below the threshold value $\nu_c(k,\gamma)$. Moreover, the axis is crossed transversally: $\partial\Re\lambda/\partial\nu < 0$. Therefore, the loss of stability occurs through a Hopf bifurcation. Numerical computations [30] reveal a rich array of unstable behaviors in the strip geometry.

It is easy to see from the definition (3.17) of the neutral stability curves that $\nu_c = 1/3$ when $k = 0$, regardless of the value of γ. Figure 1 shows that for each value of γ, there

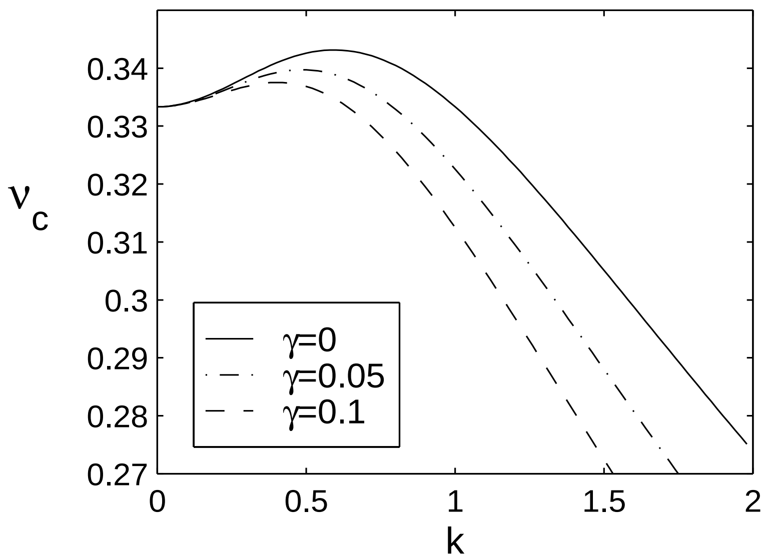

Figure 1: Neutral stability curves for various choices of surface tension parameter γ

is also a second value of k at which $\nu_c = 1/3$. For example, if $\gamma = 0$, we have $\nu_c = 1/3$ when $k = 1$, as well as when $k = 0$. In the rest of this paper, we take for simplicity $\gamma = 0$, although the subsequent calculations can accommodate $\gamma \neq 0$ with relative ease. When $\gamma = 0$, the neutral stability relation in (3.17) can be solved explicitly. Note that $\nu_c > 1/3$ for $0 < k < 1$, and $\nu_c < 1/3$ for $k > 1$ in this case.

Recall that we are interested in the discrete values $k = k_j = j\pi/L$, $j = 0, 1, 2, \dots$ that allow the normal-mode solution (1.1) to satisfy the insulated-wall and zero-contact-angle conditions (3.7). Therefore, if $L < \pi$, all modes k_j are stable for $\nu > 1/3$. Exactly one mode ($k_0 = 0$) loses stability at $\nu_c = 1/3$. Recall that $k_0 = 0$ corresponds to the dynamics with no spatial variation in the transverse direction (*i.e.*, to the one-dimensional case). If, on the other hand, $\pi < L < 2\pi$ then, as we decrease ν, the mode $k_1 = \pi/L$ loses stability prior to the flat mode, namely at a value of $\nu > 1/3$. See [38] for a detailed analysis of both of these cases $L \neq \pi$. For a similar treatment of the case in which a wavy mode loses stability first, see also [19].

If $L = \pi$, the flat mode and the mode $k_1 = 1$ both lose stability at $\nu_c = 1/3$, while the other modes remain stable. The nonlinear interaction of the flat and curvy modes $k_0 = 0$ and $k_1 = 1$ is the subject of the weakly nonlinear analysis in Section 5. Note that in this two-mode case, the Hopf bifurcation mentioned above is degenerate in the sense that two pairs of eigenvalues cross the $\Im\lambda$ axis simultaneously.

In Figure 1, observe that the addition of surface tension effects $\gamma > 0$ lowers the dispersion curve, delaying the onset of instability as we decrease ν and eliminating some unstable modes altogether (depending on the strip width L). This confirms the well-known stabilizing effect of surface tension.

4 The adjoint linear problem

To avoid secular terms in the asymptotic expansions of Section 5, we must first define and solve the adjoint of the linear problem (3.5)–(3.8). To begin, we define the inner product of u and v as

$$(u,v) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^L \int_0^\infty u(\xi,\eta,\tau)\overline{v(\xi,\eta,\tau)}\, d\eta\, d\xi\, d\tau. \tag{4.1}$$

Throughout this section, we assume that $u, v \in L^2(\Omega)$, where $\Omega = \{(\xi,\eta,\tau) | 0 \le \xi \le L,\ 0 \le \eta < \infty,\ 0 \le \tau < \infty\}$, and that u and v are bounded on Ω. Also,

$$\left.\frac{\partial u}{\partial \xi}\right|_{\xi=0,L} = 0 \quad \text{and} \quad \lim_{\eta\to\infty} u = 0. \tag{4.2}$$

To obtain solvability conditions for the perturbed problem in Section 5, we must find the null space of the adjoint operator. We define a function u to be in the null space of the adjoint operator if

$$\left(\frac{\partial v}{\partial \tau} + \mathcal{L}(v,\phi), u\right) = 0, \tag{4.3}$$

for all functions $v(\xi,\eta,\tau)$ such that

$$\mathcal{M}(v,\phi) = 0, \quad \mathcal{N}(v,\phi) = 0, \quad \left.\frac{\partial v}{\partial \xi}\right|_{\xi=0,L} = 0, \text{ and } \lim_{\eta\to\infty} v = 0, \tag{4.4}$$

and for all functions $\phi(\xi,\tau)$, bounded on $[0,L]\times[0,\infty)$ and satisfying

$$\left.\frac{\partial \phi}{\partial \xi}\right|_{\xi=0,L} = 0, \tag{4.5}$$

where $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ are as defined as in (3.9)–(3.10).

Using the definition of the inner product (4.1), integrating by parts in ξ and η, and applying the boundary conditions (4.4), the inner product in equation (4.3) can be expressed as

$$\begin{aligned}\left(\frac{\partial v}{\partial \tau} + \mathcal{L}(v,\phi), u\right) = \left(v, -\frac{\partial u}{\partial \tau} + \mathcal{L}^+ u\right) &+ \lim_{T\to\infty}\frac{1}{T}\{\mathcal{B}(\phi_\tau, \phi_{\xi\xi}, u, u_\eta) \\ &+ \text{boundary terms}\},\end{aligned} \tag{4.6}$$

where $\mathcal{L}^+ u = -u_{\xi\xi} - u_{\eta\eta} + u_\eta$, and

$$\begin{aligned}\mathcal{B}(\phi_\tau, \phi_{\xi\xi}, u, u_\eta) &= \int_0^T \int_0^L \Bigg[\int_0^\infty e^{-\eta}(\phi_\tau - \phi_{\xi\xi})\bar{u}\, d\eta \\ &- \phi_\tau \bar{u}\,|_{\eta=0} + (\nu\phi_\tau - \gamma\phi_{\xi\xi})(\bar{u} - \bar{u}_\eta)\,|_{\eta=0}\Bigg] d\xi d\tau.\end{aligned} \tag{4.7}$$

Some of the terms labeled "boundary terms" in (4.6) vanish in the limit as $T \to \infty$. Others are zero due to the conditions (4.2) on u. Still others disappear due to the last two conditions in (4.4).

In order to make the operator $-\partial_\tau + \mathcal{L}^+$ adjoint to $\partial_\tau + \mathcal{L}$, we will define its domain by the relation

$$\lim_{T\to\infty} \frac{1}{T}\mathcal{B}(\phi_\tau, \phi_{\xi\xi}, u, u_\eta) = 0. \tag{4.8}$$

Shortly we will reexpress the relation as a boundary condition.

In the meantime, making the assumption (4.8), the definition in (4.3)–(4.5) can be restated as follows: a function u is in the null space of the adjoint operator if

$$\left(v, -\frac{\partial u}{\partial \tau} + \mathcal{L}^+ u\right) = 0$$

for all $v(\xi, \eta, \tau)$ satisfying the boundary conditions (4.4). Therefore, u will be in the null space of the adjoint operator if

$$-\frac{\partial u}{\partial \tau} + \mathcal{L}^+ u = 0. \tag{4.9}$$

We can then solve the partial differential equation (4.9) by separation of variables, subject to the conditions (4.2) on u, to get

$$u = e^{\mu\tau}\cos(k_j\xi)\tilde{h}(\eta;\mu), \tag{4.10}$$

where

$$\tilde{h}'' - \tilde{h}' - (k_j^2 - \mu)\tilde{h} = 0, \tag{4.11}$$

$$\lim_{\eta\to\infty} \tilde{h}(\eta;\mu) = 0, \tag{4.12}$$

and $k_j = j\pi/L$, $j = 0, 1, 2, \ldots$. In order for u to be bounded on Ω, we also need $\Re\mu < 0$.

To translate (4.8) into a boundary condition on $\tilde{h}(\eta;\mu)$, we first substitute u of the form (4.10) into the expression in (4.7) for $\mathcal{B}$, integrate by parts with respect to τ and ξ, divide by T, and take a limit to get

$$\begin{aligned} &\lim_{T\to\infty} \tfrac{1}{T} \quad \mathcal{B}(\phi_\tau, \phi_{\xi\xi}, u, u_\eta) = \\ &\lim_{T\to\infty} \{\tfrac{1}{T} \quad B(h,\mu,k_j,\gamma)\int_0^L\int_0^T \phi(\xi,\tau)e^{\bar{\mu}\tau}\cos(k_j\xi)\,d\tau d\xi + \text{boundary terms}\}, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} B(h,\mu,k_j,\gamma) &= \left(\bar{\mu}(1-\nu) + k_j^2\gamma\right)\bar{\tilde{h}}(0;\mu) \\ &+ (\bar{\mu}\nu - k_j^2\gamma)\bar{\tilde{h}}'(0;\mu) \\ &+ \int_0^\infty e^{-\eta}(k_j^2 - \bar{\mu})\bar{\tilde{h}}(\eta;\mu)\,d\eta. \end{aligned} \tag{4.14}$$

Some of the terms labeled "boundary terms" in (4.13) vanish in the limit as $T \to \infty$. The others are zero due to condition (4.5) on ϕ.

To satisfy condition (4.8), it is sufficient that $B(h,\mu,k_j,\gamma) = 0$. To eliminate the integral from (4.14), we multiply the equation (4.11) by $e^{-\eta}$ and integrate by parts to get

$$\int_0^\infty e^{-\eta}(k_j^2 - \bar{\mu})\bar{\tilde{h}}(\eta;\mu)\,d\eta = -\bar{\tilde{h}}'(0;\mu).$$

Substituting the above right-hand side into equation into (4.14) for the integral and taking the complex conjugate lets us write $B(h, \mu, k_j, \gamma) = 0$ as

$$\left(\mu(1-\nu) + k_j^2\gamma\right)\tilde{h}(0;\mu) + (\mu\nu - k_j^2\gamma - 1)\tilde{h}'(0;\mu) = 0. \tag{4.15}$$

This condition, together with the differential equation (4.11) and the condition as $\eta \to \infty$ (4.12), constitute the adjoint eigenvalue problem.

The general solution to the differential equation (4.11) is

$$\tilde{h}(\eta;\mu) = c_1 \exp\left(\frac{1+\sqrt{1+4(k_j^2-\mu)}}{2}\eta\right) + c_2 \exp\left(\frac{1-\sqrt{1+4(k_j^2-\mu)}}{2}\eta\right).$$

Setting $c_1 = 0$ and applying condition (4.15) reproduces the dispersion relation (3.15) for $\mu = -\bar{\lambda}_j$. Taking the value of the arbitrary normalization constant c_2 to be 1 gives the adjoint eigenfunction

$$h(\eta;\lambda_j) = \exp\left(\left(1-\sqrt{1+4(k_j^2+\bar{\lambda}_j)}\right)\frac{\eta}{2}\right), \tag{4.16}$$

where $h(\eta;\lambda_j) = \tilde{h}(\eta;-\bar{\lambda}_j)$.

A solution to the adjoint of the linearized problem is

$$u(\xi,\eta,\tau;\lambda_j) = e^{-\bar{\lambda}_j\tau}\cos(k_j\xi)h(\eta;\lambda_j). \tag{4.17}$$

Note that if $j = 0$ then $k_j = 0$, and $(\mu, \tilde{h}(\eta;\mu)) = (0,1)$ is also an adjoint eigenvalue-eigenfunction pair. Thus

$$u(\xi,\eta,\tau;0) = 1 \tag{4.18}$$

is also an adjoint solution. The presence of this trivial eigenmode is due to the invariance of the original problem with respect to a shift in the η direction. Only the front velocity, not its position, enters the formulation of the direct problem.

5 Weakly nonlinear analysis

In this section we first consider the geometry in which the strip has width $L = \pi$ for the solid combustion application ($\gamma = 0$). As the material parameter ν drops below the critical value $\nu_c = 1/3$, the two modes with wave numbers

$$k_0 = 0, \quad k_1 = 1 \tag{5.1}$$

lose stability simultaneously. We will indicate how the problem simplifies when the strip width L is such that a single mode loses stability and state results for that case, as well.

We begin by considering a small deviation from the neutral stability curve in the unstable direction, namely

$$\epsilon^2 = \nu_c - \nu = \frac{1}{3} - \nu. \tag{5.2}$$

This choice of the parameter ϵ allows for the possibility of a Hopf bifurcation where the magnitude of the solution is on the order of the square root of the bifurcation parameter.

The eigenvalues in the weakly nonlinear regime can be found by expanding λ_j in a Taylor series about $\nu = 1/3$. Then $\lambda_j = \lambda_j(\nu)$ has the form

$$\lambda_j(\nu) = i\omega_j - \epsilon^2 \left.\frac{\partial \lambda_j}{\partial \nu}\right|_{\nu=\nu_c} + O(\epsilon^4). \tag{5.3}$$

To find the neutrally stable eigenvalues $i\omega_j$, we evaluate the dispersion relation (3.15) at $\nu = \nu_c = 1/3$ (implying $\lambda_j = i\omega_j$). Solving the real or imaginary part gives

$$\omega_0 = \sqrt{3}, \quad \omega_1 = 3. \tag{5.4}$$

Note that $g(\eta; i\omega_j, \nu_c)$ is the eigenfunction corresponding to the neutrally stable eigenvalue $i\omega_j$. (See (3.16).) The eigenvalue-eigenfunction pair appears in our ansatz (1.2).

As for the perturbation of the neutrally stable eigenvalue, the coefficient of ϵ^2 in (5.3) is calculated by differentiating the dispersion relation (3.15) with respect to ν and evaluating at $\nu = 1/3$ (again implying $\lambda_j = i\omega_j$) to get

$$-\left.\frac{\partial \lambda_0}{\partial \nu}\right|_{\nu=\nu_c} = \frac{3}{2}(9 + \sqrt{3}i) = \chi_0, \tag{5.5}$$

$$-\left.\frac{\partial \lambda_1}{\partial \nu}\right|_{\nu=\nu_c} = \frac{27}{2} + 9i = \chi_1. \tag{5.6}$$

The quantities χ_0 and χ_1 in (5.5) and (5.6) will appear in the Landau-Stuart equations (1.4) governing the dynamics of the weakly unstable modes.

5.1 The asymptotic strategy

The goal of this section is to find a solution to the nonlinear problem (3.1)–(3.4) of the form (1.2). The solution involves a combination of modulations of neutrally stable solutions to the linearized problem, corresponding to $k_0 = 0$ and $k_1 = 1$.

The asymptotic strategy is to insert the expansions (1.2), $\nu = 1/3 - \epsilon^2$, and the Taylor series for the kinetic function $K(V)$ about $V = 1$ into the problem (3.1)–(3.4). (Note that the Taylor expansion of K about the basic solution ($V = 1$) contains terms that are nonlinear in σ and ν.) Because we have introduced the independent time scales

$$t_0 = \tau, \quad t_1 = \epsilon\tau, \quad t_2 = \epsilon^2\tau,$$

we get $\partial/\partial\tau = \partial/\partial t_0 + \epsilon\, \partial/\partial t_1 + \epsilon^2\, \partial/\partial t_2$. These substitutions produce the partial differential equation

$$\begin{aligned} u_\tau - u_{\xi\xi} + 2u_{\xi\eta} f_\xi - u_\eta(f_\tau - f_{\xi\xi}) - u_{\eta\eta}(f_\xi^2 + 1) = \\ + \quad \epsilon^2 \left\{ \frac{\partial w_2}{\partial t_0} + \mathcal{L}(w_2, \phi_2) + \frac{\partial}{\partial t_1}\Big(w_1 + e^{-\eta}\phi_1\Big) - N_1(w_1, \phi_1) \right\} \end{aligned} \tag{5.7}$$

$$+ \quad \epsilon^3 \Big\{ \frac{\partial w_3}{\partial t_0} + \mathcal{L}(w_3, \phi_3) + \frac{\partial}{\partial t_1}\Big(w_2 + e^{-\eta}\phi_2\Big) + \frac{\partial}{\partial t_2}\Big(w_1 + e^{-\eta}\phi_1\Big)$$
$$- \quad N_2(w_1, \phi_1, w_2, \phi_2)\Big\} + \mathrm{CC} + O(\epsilon^4) = 0.$$

The conditions on the reaction front are expanded as

$$u(\xi, 0, \tau) - 1 - \nu K(V) = \epsilon^2 \left\{ \mathcal{M}(w_2, \phi_2) - \frac{1}{3}\frac{\partial \phi_1}{\partial t_1} - N_3(\phi_1) \right\} \tag{5.8}$$
$$+ \quad \epsilon^3 \left\{ \mathcal{M}(w_3, \phi_3) - \frac{1}{3}\left(\frac{\partial \phi_2}{\partial t_1} + \frac{\partial \phi_1}{\partial t_2}\right) + \frac{\partial \phi_1}{\partial t_0} - N_4(\phi_1, \phi_2) \right\} + \mathrm{CC} + O(\epsilon^4) = 0,$$

$$f_\xi u_\xi \Big|_{\eta=0} - (f_\xi^2 + 1) u_\eta \Big|_{\eta=0} - f_\tau = \epsilon^2 \left\{ \mathcal{N}(w_2, \phi_2) + \frac{\partial \phi_2}{\partial t_1} - N_5(\phi_1, w_1) \right\} \tag{5.9}$$
$$+ \quad \epsilon^3 \left\{ \mathcal{N}(w_3, \phi_3) + \frac{\partial \phi_2}{\partial t_1} + \frac{\partial \phi_1}{\partial t_2} - N_6(w_1, \phi_1, w_2, \phi_2) \right\} + \mathrm{CC} + O(\epsilon^4) = 0,$$

where w_j and ϕ_j are the $O(\epsilon^j)$ perturbations of u and f respectively, given in (1.2). The operators $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ are as defined in (3.9)–(3.10), except that we use now use $\mathcal{M}(w, \phi)$ to mean $\mathcal{M}(w, \phi)|_{\nu=1/3}$ (and we continue to use this shorthand throughout the remainder of the article). Terms labeled N_j are nonlinear functions of the arguments indicated. (See Appendix A.)

Equating like powers of ϵ results in subproblems for the terms in the perturbation expansions (1.2), subject to solvability conditions on the amplitudes A_0, A_1, and B. Note that no $O(1)$ terms appear in the expanded problem (5.7)–(5.9) since in (1.2) we took the temperature-interface pair (u, f) perturbed about $(e^{-\eta}, t_0)$, a solution to the nonlinear problem (3.1)–(3.4). Similarly, no $O(\epsilon)$ terms appear in the expanded problem (5.7)–(5.9). The linearized problem (3.5)–(3.8) with $\nu = \nu_c$ is satisfied identically by the $O(\epsilon)$ terms in the expansions (1.2) of u, f, namely

$$w_1 = \sum_{j=0}^{1} A_j(t_1, t_2) e^{i\omega_j t_0} \cos(k_j \xi) g(\eta; i\omega_j, \nu_c), \tag{5.10}$$
$$\phi_1 = \sum_{j=0}^{1} A_j(t_1, t_2) e^{i\omega_j t_0} \cos(k_j \xi).$$

We will examine the $O(\epsilon^2)$ and $O(\epsilon^3)$ subproblems below. The solvability condition on the $O(\epsilon^3)$ problem will lead to the system (1.4) that determines the dynamics of the unstable modes.

5.2 The $O(\epsilon^2)$ problem

Solvability conditions for the $O(\epsilon^2)$ problem will show that the complex amplitudes A_0 and A_1 depend on the slowest time scale t_2 only, will give an expression for B in terms of A_0 and A_1, and will give the forms of w_2 and ϕ_2 with A_0 and A_1 still unknown.

Substituting w_1 and ϕ_1 as given in (5.10), the $O(\epsilon^2)$ subproblem included in (5.7)–(5.9) can be written

$$\frac{\partial w_2}{\partial t_0} + \mathcal{L}(w_2, \phi_2) = \Bigg(- \sum_{j=0}^{1} \frac{\partial A_j}{\partial t_1} e^{i\omega_j t_0} \left(g(\eta; i\omega_j, \nu_c) + e^{-\eta} \right) \tag{5.11}$$

$$+ \quad \hat{R}_2(\xi,\eta,\mathbf{t}) + \mathrm{CC}\Big) - \frac{\partial B}{\partial t_1} e^{-\eta} = Q_2(\xi,\eta,\mathbf{t}),$$

$$\mathcal{M}(w_2,\phi_2) = \frac{1}{3}\left\{\left(\sum_{j=0}^{1} \frac{\partial A_j}{\partial t_1} e^{i\omega_j t_0} \cos(k_j\xi) \right.\right. \tag{5.12}$$

$$\left.\left. + \quad \hat{a}_2(\xi,\mathbf{t}) + \mathrm{CC}\right) + \frac{\partial B}{\partial t_1}\right\} = \alpha_2(\xi,\mathbf{t}),$$

$$\mathcal{N}(w_2,\phi_2) = \left(-\sum_{j=0}^{1} \frac{\partial A_j}{\partial t_1} e^{i\omega_j t_0} \cos(k_j\xi)\right. \tag{5.13}$$

$$\left. + \quad \hat{b}_2(\xi,\mathbf{t}) + \mathrm{CC}\right) - \frac{\partial B}{\partial t_1} = \beta_2(\xi,\mathbf{t}),$$

where $\mathbf{t} = (t_0, t_1, t_2)$, and we have named the right-hand sides above as Q_2, α_2, and β_2. $\hat{R}_2(\xi,\eta,\mathbf{t})$ in (5.11) has terms of the form

$$A_{l_1}(\Re A_{l_2} + n\Im A_{l_2}) e^{i(\omega_{l_1}+n\omega_{l_2})t_0} \cos(m\xi)\hat{R}_{l_1,l_2,n,m}(\eta), \tag{5.14}$$

where $(l_1, l_2, n, m) \in S$,

$$S = \{(0,0,1,0),(0,0,-1,0),(0,1,1,1),(0,1,-1,1),(1,0,-1,1), \\ (1,1,1,2),(1,1,1,0),(1,1,-1,2),(1,1,-1,0)\}. \tag{5.15}$$

The terms $\hat{a}_2(\xi,\mathbf{t})$ and $\hat{b}_2(\xi,\mathbf{t})$ in (5.12) and (5.13) have the same form as $\hat{R}_2(\xi,\eta,\mathbf{t})$ given in (5.14), except with $\hat{R}_{l_1,l_2,n,m}(\eta)$ replaced by $\hat{F}_{l_1,l_2,n,m}$ and $\hat{G}_{l_1,l_2,n,m}$, respectively. The coefficients are

$$\begin{aligned}
\hat{R}_{l_1,l_2,n,m}(\eta) &= \frac{1}{2}\Big\{g'(\eta; i\omega_{l_1}, \nu_c)\Big[(2 + l_1 l_2(2m-3))\ k_{l_2}^2 + in\ (2 - l_1 l_2)\ \omega_{l_2}\Big] \\
&+ \ i(l_2 - l_1)(n+m) g'(\eta; i\omega_1, \nu_c)\sqrt{3} + l_1 l_2 (-1)^{m/2} e^{-\eta}\Big\}, \\
\hat{F}_{l_1,l_2,n,m} &= \frac{1}{4}\Big\{(l_1 l_2 - 2)\ nK''(1)|_{\nu=1/3}\ \omega_{l_1}\omega_{l_2} - l_1 l_2 (-1)^{m/2}\Big\}, \\
\hat{G}_{l_1,l_2,n,m} &= \frac{1}{2} l_1 l_2 (-1)^{m/2}(1+i).
\end{aligned}$$

Recall that $g(\eta;\lambda,\nu)$ is the eigenfunction (3.16).

To derive the Fredholm solvability conditions, we first change into a new variable u_2 to produce homogeneous boundary conditions

$$\mathcal{M}(u_2,\phi_2) = \mathcal{N}(u_2,\phi_2) = 0. \tag{5.16}$$

In particular, let $u_2 = w_2 - \alpha_2\mathcal{S}(\eta) - \beta_2\mathcal{T}(\eta)$, where α_2 and β_2 are the right-hand sides of equations (5.12) and (5.13) respectively, and $\mathcal{S}(\eta)$ and $\mathcal{T}(\eta)$ satisfy the boundary conditions

$$\mathcal{S}(0) = 1, \quad \mathcal{S}'(0) = 0, \quad \mathcal{T}(0) = 0, \quad \mathcal{T}'(0) = 1, \quad \lim_{\eta\to\infty}\mathcal{S}(\eta) = \lim_{\eta\to\infty}\mathcal{T}(\eta) = 0. \tag{5.17}$$

We could take for example $\mathcal{S}(\eta) = (1+\eta)e^{-\eta}$, $\mathcal{T}(\eta) = \eta e^{-\eta}$.

The $O(\epsilon^2)$ partial differential equation (5.11) in the new variable is

$$\frac{\partial u_2}{\partial t_0} + \mathcal{L}(u_2, \phi_2) = Q_2(\eta, \xi, \mathbf{t}) - \left\{ \frac{\partial \alpha_2}{\partial t_0}\mathcal{S}(\eta) + \frac{\partial \beta_2}{\partial t_0}\mathcal{T}(\eta) \right. \qquad (5.18)$$
$$\left. + \mathcal{L}\left(\alpha_2 \mathcal{S}(\eta), 0\right) + \mathcal{L}\left(\beta_2 \mathcal{T}(\eta), 0\right) \right\} = P_2(\xi, \eta, \mathbf{t}).$$

By the Fredholm alternative theorem, equation (5.18) together with the homogeneous boundary conditions (5.16) has a solution if and only if the inhomogeneity is orthogonal to the null space of the adjoint operator. Therefore the right-hand side named $P_2(\xi, \eta, \mathbf{t})$ in the partial differential equation (5.18) must be orthogonal to the adjoint solutions u_0 and u_1 in (4.17). The integral in the orthogonality condition $(P_2(\xi, \eta, \mathbf{t}), u_j(\xi, \eta, t_0)) = 0$ can be simplified by noting that since $\lambda_j = i\omega_j + \epsilon^2 \chi_j$, $j = 0, 1$, and $\epsilon^2 \tau = t_2$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^{(-\lambda_j + i\omega_{l_1} + in\omega_{l_2})\tau}\, d\tau = \begin{cases} e^{-\chi_j t_2} \neq 0 & \text{if } l_1 = j \text{ and } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Exploiting also the orthogonality of cosines with differing wave numbers, integrating by parts in η, using our knowledge of the initial and limiting values of $\mathcal{S}$, $\mathcal{T}$, and their derivatives, and neglecting higher-order terms, the solvability conditions can be expressed as

$$\frac{\partial A_j}{\partial t_1} \left\{ \int_0^\infty \left(g(\eta; i\omega_j, \nu_c) + e^{-\eta} \right) \bar{h}_j(\eta)\, d\eta + \frac{2}{3}\bar{h}_j(0) + \bar{h}_j'(0) \right\} = 0, \ j = 0, 1,$$

where $h_j(\eta)$ is shorthand for $h(\eta; i\omega_j)$. (See 4.16.) Because the second factor is nonzero,

$$\frac{\partial A_j}{\partial t_1} = 0, \quad j = 0, 1. \qquad (5.19)$$

That is, the amplitudes depend on the the slowest time scale t_2 only.

Fredholm's Alternative requires that the right-hand side $P_2(\xi, \eta, \mathbf{t})$ of equation (5.18) also must be orthogonal to the adjoint solution $u(\xi, \eta, t_0) = 1$. Using the information summarized above, the solvability condition $(P_2(\xi, \eta, \mathbf{t}), 1) = 0$ can be expressed as

$$\frac{\partial B}{\partial t_1} = -2 \sum_{j=0}^{1} A_j \bar{A}_j r_j, \quad r_j = -\frac{1}{4}(2-j)[k_j^2 - \omega_j^2(2 + K''(1)|_{\nu=1/3})]. \qquad (5.20)$$

Applying the conditions (5.19) and (5.20) puts the $O(\epsilon^2)$ problem in solvable form. In particular, we get (5.11)–(5.13) with $\partial A_j/\partial t_1$ terms no longer appearing and $\partial B/\partial t_1$ terms absorbed into $\hat{R}_2$, $\hat{a}_2$, and $\hat{b}_2$ to produce R_2, a_2, and b_2, respectively. R_2 terms are as defined in (5.14) except with the hat dropped from $\hat{R}_{l_1,l_2,n,m}(\eta)$. The analogous statements hold for a_2 and b_2. The coefficients in the three expressions are

$$R_{l_1,l_2,n,m}(\eta) \quad = \quad \hat{R}_{l_1,l_2,n,m}(\eta) + \frac{1}{24}\prod_{j=0}^{3}(n+m-j)\, r_{l_1} e^{-\eta},$$

$$F_{l_1,l_2,n,m} = \hat{F}_{l_1,l_2,n,m} - \frac{1}{24}\prod_{j=0}^{3}(n+m-j)\, r_{l_1},$$

$$G_{l_1,l_2,n,m} = \hat{G}_{j,j,n,m} + \frac{1}{24}\prod_{j=0}^{3}(n+m-j)\, r_{l_1},$$

for $(l_1, l_2, n, m) \in S$ and r_0, r_1 given in (5.20).

The solution (w_2, ϕ_2) to the solvable second-order problem has the same form as the inhomogeneity $R_2(\xi, \eta, \mathbf{t})$ given in (5.14), with $\hat{R}_{l_1,l_2,n,m}(\eta)$ replaced by $g_{l_1,l_2,n,m}(\eta)$ in w_2 and by $C_{l_1,l_2,n,m}$ in ϕ_2. The functions $g_{l_1,l_2,n,m}(\eta)$ and $C_{l_1,l_2,n,m}$ for $(l_1, l_2, n, m) \in S$ are given in Appendix B. They satisfy the initial value problems

$$\begin{aligned} g''_{l_1,l_2,n,m} + g'_{l_1,l_2,n,m} - \Big(m^2 + i(\omega_{l_1} + n\omega_{l_2})\Big)\, g_{l_1,l_2,n,m}(\eta) \\ = \Big(m^2 + i(\omega_{l_1} + n\omega_{l_2})\Big)\, C_{l_1,l_2,n,m} e^{-\eta} - R_{l_1,l_2,n,m}(\eta), \end{aligned} \tag{5.21}$$

$$g_{l_1,l_2,n,m}(0) = \frac{1}{3}\left[F_{l_1,l_2,n,m} + i(\omega_{l_1} + n\omega_{l_2})C_{l_1,l_2,n,m}\right] \tag{5.22}$$

$$g'_{l_1,l_2,n,m}(0) = G_{l_1,l_2,n,m} - i(\omega_{l_1} + n\omega l_2)C_{l_1,l_2,n,m}, \tag{5.23}$$

where $(l_1, l_2, n, m) \in S$.

5.3 The $O(\epsilon^3)$ problem

Solvability conditions for the $O(\epsilon^3)$ problem will lead to a system of ordinary differential equations for the slowly-varying amplitudes. When we substitute w_1, ϕ_1, w_2, and ϕ_2, which we know in terms of A_0, A_1, and B, into (5.7)–(5.9), the $O(\epsilon^3)$ problem has the form

$$\frac{\partial w_3}{\partial t_0} + \mathcal{L}(w_3, \phi_3) = Q_3(\xi, \eta, \mathbf{t}), \tag{5.24}$$

$$\mathcal{M}(w_3, \phi_3) = \alpha_3(\xi, \mathbf{t}),\ \mathcal{N}(w_3, \phi_3) = \beta_3(\xi, \mathbf{t}). \tag{5.25}$$

To derive solvability conditions, we proceed as in the previous section, changing into a new variable $u_3 = w_3 - \alpha_3\mathcal{S}(\eta) - \beta_3\mathcal{T}(\eta)$ to produce the homogeneous boundary conditions $\mathcal{M}(u_3, \phi_3) = \mathcal{N}(u_3, \phi_3) = 0$. ($\mathcal{S}(\eta)$ and $\mathcal{T}(\eta)$ satisfy the conditions (5.17).)

By the Fredholm Alternative Theorem, the $O(\epsilon^3)$ partial differential equation in ϕ_3 and the new variable u_3 has a solution, subject to the homogeneous boundary conditions, only if its right-hand side

$$\mathrm{RHS} = Q_3(\eta, \xi, \mathbf{t}) - \left\{\frac{\partial \alpha_3}{\partial t_0}\mathcal{S}(\eta) + \frac{\partial \beta_3}{\partial t_0}\mathcal{T}(\eta) + \mathcal{L}\left(\alpha_3\mathcal{S}(\eta), 0\right) + \mathcal{L}\left(\beta_3\mathcal{T}(\eta), 0\right)\right\} \tag{5.26}$$

is orthogonal to the adjoint solutions $u_0(\xi, \eta, \tau_0)$, $u_1(\xi, \eta, \tau_0)$ given in (4.17). Due to the orthogonality of cosines and of exponentials discussed in the previous section, many terms in the inner product are zero. We need only note that the right-hand sides in (5.24)–(5.25) have the forms

$$\begin{aligned} Q_3(\xi,\eta,\mathbf{t}) &= \sum_{j=0}^{1} e^{i\omega_j t_0}\cos(k_j\xi)\Big\{ -\frac{\partial A_j}{\partial t_2}\left(g(\eta;i\omega_j,\nu_c)+e^{-\eta}\right) + \\ &\quad + A_j^2\bar{A}_j\mathcal{R}_{j,1}(\eta) + A_jA_{1-j}\bar{A}_{1-j}\mathcal{R}_{j,2}(\eta)\Big\} + \\ &\quad + \mathrm{CC} + \text{other terms}, \end{aligned} \tag{5.27}$$

$$\begin{aligned} \alpha_3(\xi,\mathbf{t}) &= \sum_{j=0}^{1} e^{i\omega_j t_0}\cos(k_j\xi)\Big\{\frac{1}{3}\frac{\partial A_j}{\partial t_2} - i\omega_jA_j + \\ &\quad + A_j^2\bar{A}_j\mathcal{F}_{j,1} + A_jA_{1-j}\bar{A}_{1-j}\mathcal{F}_{j,2}\Big\} + \\ &\quad + \mathrm{CC} + \text{other terms}, \end{aligned} \tag{5.28}$$

$$\begin{aligned} \beta_3(\xi,\mathbf{t}) &= \sum_{j=0}^{1} e^{i\omega_j t_0}\cos(k_j\xi)\Big\{ -\frac{\partial A_j}{\partial t_2} + \\ &\quad + A_j^2\bar{A}_j\mathcal{G}_{j,1} + A_jA_{1-j}\bar{A}_{1-j}\mathcal{G}_{j,2}\Big\} + \\ &\quad + \mathrm{CC} + \text{other terms}. \end{aligned} \tag{5.29}$$

$\mathcal{R}_{j,n}(\eta)$, $\mathcal{F}_{j,n}$, and $\mathcal{G}_{j,n}$ are given in Appendix C for $j=0,1$, $n=1,2$.

Doing the integration $(\text{RHS}, u_j(\xi,\eta,t_0))=0$, for RHS given in (5.26), yields the system of Landau-Stuart equations as promised in (1.4), namely

$$\frac{dA_j}{dt_2} = \chi_jA_j + \beta_{j,1}A_j^2\bar{A}_j + \beta_{j,2}A_jA_{1-j}\bar{A}_{1-j}, \quad j=0,1, \tag{5.30}$$

where χ_0 and χ_1 are given in (5.5) and (5.6), and

$$\beta_{j,n} = \frac{\frac{1}{3}\mathcal{F}_{j,n}\mathcal{U}_j - \mathcal{G}_{j,n} + \int_0^\infty \mathcal{R}_{j,n}(\eta)\bar{h}_j(\eta)\,d\eta}{-\frac{1}{3}\mathcal{U}_j - 1 + \int_0^\infty \left(g(\eta;i\omega_j,\nu_c)+e^{-\eta}\right)\bar{h}_j(\eta)\,d\eta}, \quad j=0,1, \; n=1,2. \tag{5.31}$$

Recall that $g(\eta;\lambda,\nu)$ is given in (3.16), $h_j(\eta)$ is the adjoint eigenfunction (4.16) when $\lambda_j = i\omega_j$, and that $\mathcal{R}_{j,n}(\eta)$, $\mathcal{F}_{j,n}$, and $\mathcal{G}_{j,n}$ are given in Appendix C. (Note that because the $\mathcal{F}_{j,n}$ depend on derivatives of the kinetics function (2.8), namely $K^{(i)}(V)|_{\nu=1/3}$ for $i=2,3$, it follows that the $\beta_{j,n}$ are functions of the kinetics parameter σ. In fact, they turn out to be quadratics in σ.) In the expression (5.31) for $\beta_{j,n}$, $\mathcal{U}_j$ is $-\bar{h}_j(0)+\bar{h}_j'(0)$, so $\mathcal{U}_0 = -(3+i\sqrt{3})/2$, and $\mathcal{U}_1 = -2-i$. The integrals in (5.31) are also straightforward to compute. (The author calculated all six using Maple to handle the many terms.)

6 Asymptotic dynamics

The Landau-Stuart equations (5.30) completely determine the dynamics of the weakly unstable modes

$$A_j(t_2)e^{i\omega_j t_0}\cos k_j\xi, \quad j=0,1, \tag{6.1}$$

subject to both nonlinear interaction and self-interaction. The competition between the modes depends on the relationships among the coefficients of the system in (5.30) and is determined by the kinetics function $K(V)$, in our case given in (2.8). We characterize the outcome of the competition for all values of the nonlinear kinetics parameter σ.

We focus here on the magnitudes of the complex amplitudes. Multiplying the jth equation in (5.30) by $\bar{A}_j$ and adding the resulting equation to its complex-conjugate, yields the system in $x_j = |A_j|^2$, namely

$$\frac{dx_j}{d\tau_2} = a_j x_j + b_{j,1}\, x_j^2 + b_{j,2}\, x_j x_{1-j}, \tag{6.2}$$

where $a_j = 2\Re\chi_j$, and $b_{j,n} = 2\Re\beta_{j,n}$, $j = 0, 1$, $n = 1, 2$.

The previous section sketched the calculation of the coefficients $\beta_{j,n}$ as quadratics in the kinetics parameter σ, and χ_0 and χ_1 are given in (5.5) and (5.6). The coefficients a_j and $b_{j,n}$ in (6.2) can be computed accordingly, and one can determine positions of the critical points of the system and analyze their stability.

For $0 < \sigma < 0.52$ the system (6.2) has four critical points in the first quadrant: two saddles on the x_0- and x_1-axes, an unstable node at the origin, and an asymptotically stable node in the interior. Thus, for this range of the kinetics parameter σ, the amplitude equations predict an asymptotic regime of mode coexistence, no matter what the initial data are. A typical direction field is given in Figure 2.

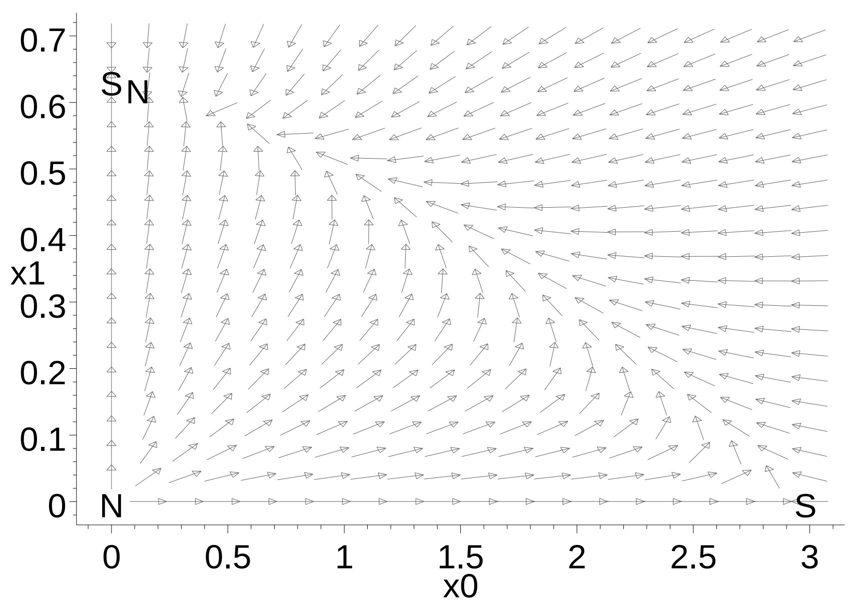

Figure 2: A direction field for the amplitude equations in $x_0 = |A_0|^2$ and $x_1 = |A_1|^2$ for $\sigma = 0.45$. S indicates a saddle, N a node.

At approximately $\sigma = 0.52$ there is an exchange of stability. (We note that this estimate of the critical value of σ is more accurate and slightly smaller than the one given in earlier work [30].) Mode coexistence breaks down at this value of the kinetics parameter, as the critical point in the interior of the first quadrant moves into the non-physical second quadrant. The saddle point on the x_0-axis simultaneously becomes an asymptotically stable node. Since $x_1 = 0$ corresponds to the absence of the "curvy" mode, the asymptotics reveal flat front propagation as σ exceeds 0.52. Flat front propagation dominates for the remaining physical values of σ ($0.52 < \sigma < 1$), no matter what the initial data are. A typical direction field is given in Figure 3.

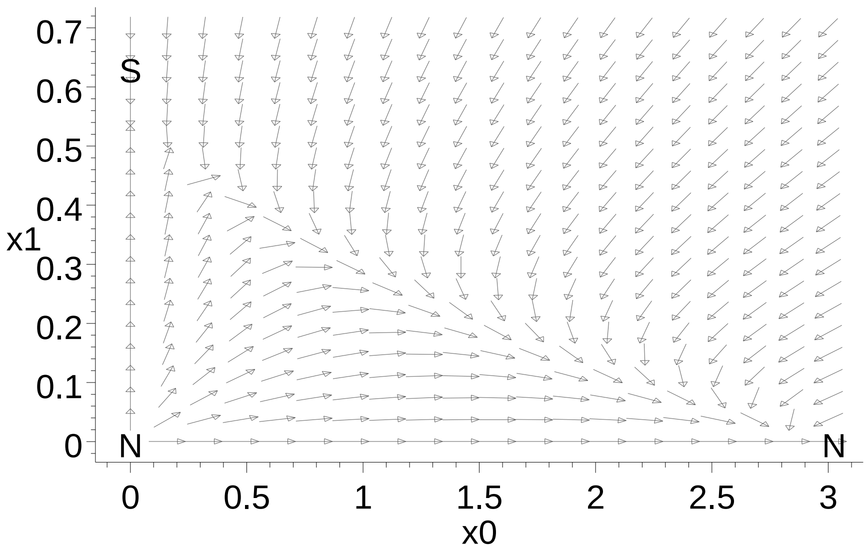

Figure 3: A direction field for $\sigma = 0.65$. S indicates a saddle, N a node.

These results for the asymptotic dynamics provide a guide for numerical computations on the free boundary problem (3.1)–(3.4). Simulations in [30] illustrate that the qualitative character of the dynamics changes at the critical value $\sigma \approx 0.52$.

Let us comment on the dynamics for strip widths $L \neq \pi$. For $L < \pi$, the first mode to become linearly unstable as we decrease ν is the flat mode $k_0 = 0$. Note that this case can be viewed as a reduction to a one-dimensional geometry. In such a case, we are interested in the self-interactions of the weakly unstable mode (6.1) for $j = 0$ only. The front behavior is determined by the dynamics of the single Landau-Stuart equation

$$\frac{dA_0}{dt_2} = \chi_j A_0 + \beta_{0,1} A_0^2 \bar{A}_0,$$

having no interaction term. We can associate with this equation an equation in $x = |A_0|^2$, as we did for the two-mode case above. In [38], we showed that for a certain choice of kinetics, the ordinary differential equation in x has a limit cycle only in a particular interval of the kinetics parameter. Computations [26] on the free boundary problem in that interval exhibit periodic behavior; computations outside do not.

Analogously, [38] also contains a complete analysis for strip width $L \in (\pi, 2\pi)$ for a certain choice of kinetics function. Self-interactions of the first linearly unstable mode $k = \pi/L$ produce periodic solutions only within a particular interval of the kinetics parameter.

7 Conclusions

In this paper we have performed a weakly nonlinear analysis of sharp reaction fronts that move along a two-dimensional semi-infinite strip with insulated edges, focusing on a strip of width π. The asymptotic analysis led to the derivation of Landau-Stuart equations for the two slowly-varying amplitudes of the two weakly linearly unstable modes: one flat and one wavy. The ordinary differential equations resulted from the application of a solvability condition to the $O(\epsilon^3)$ problem. By analyzing the dynamics of the Landau-Stuart system, we determined that if the constant ratio $0 < \sigma < 1$ of the fresh temperature to the reacted temperature exceeds approximately 0.52, then flat propagation dominates, regardless of initial conditions. Otherwise the two modes will always coexist for all time.

A Nonlinear functions in the expanded problem

The nonlinear functions in the expanded problem (5.7)–(5.9) are

$$
\begin{aligned}
N_1(w_1, \phi_1) &= \frac{\partial w_1}{\partial \eta}\left(\frac{\partial \phi_1}{\partial t_0} - \frac{\partial^2 \phi_1}{\partial \xi^2}\right) - 2\frac{\partial \phi_1}{\partial \xi}\frac{\partial^2 w_1}{\partial \xi \partial \eta} + e^{-\eta}\left(\frac{\partial \phi_1}{\partial \xi}\right)^2, \\
N_2(w_1, \phi_1, w_2, \phi_2) &= 2e^{-\eta}\frac{\partial \phi_1}{\partial \xi}\frac{\partial \phi_2}{\partial \xi} + \frac{\partial w_1}{\partial \eta}\left(\frac{\partial \phi_2}{\partial t_0} + \frac{\partial \phi_1}{\partial t_1} - \frac{\partial^2 \phi_2}{\partial \xi^2}\right) \\
&+ \frac{\partial w_2}{\partial \eta}\left(\frac{\partial \phi_1}{\partial t_0} - \frac{\partial^2 \phi_1}{\partial \xi^2}\right) - 2\left(\frac{\partial \phi_1}{\partial \xi}\frac{\partial^2 w_2}{\partial \xi \partial \eta} + \frac{\partial \phi_2}{\partial \xi}\frac{\partial^2 w_1}{\partial \xi \partial \eta}\right) \\
&+ \left(\frac{\partial \phi_1}{\partial \xi}\right)^2 \frac{\partial^2 w_1}{\partial \eta^2}, \\
N_3(\phi_1) &= \frac{1}{6}\left(K''(1)|_{\nu=1/3}\left(\frac{\partial \phi_1}{\partial t_0}\right)^2 - \left(\frac{\partial \phi_1}{\partial \xi}\right)^2\right), \\
N_4(\phi_1, \phi_2) &= \frac{1}{3}\left\{-\frac{1}{2}\left(\frac{\partial \phi_1}{\partial \xi}\right)^2 \frac{\partial \phi_1}{\partial t_0} - \frac{\partial \phi_1}{\partial \xi}\frac{\partial \phi_2}{\partial \xi}\right. \\
&+ K''(1)|_{\nu=1/3}\frac{\partial \phi_1}{\partial t_0}\left(\frac{\partial \phi_2}{\partial t_0} + \frac{\partial \phi_1}{\partial t_1} - \frac{1}{2}\left(\frac{\partial \phi_1}{\partial \xi}\right)^2\right) \\
&+ \left.\frac{1}{6}K'''(1)|_{\nu=1/3}\left(\frac{\partial \phi_1}{\partial t_0}\right)^3\right\},
\end{aligned}
$$

$$
\begin{aligned}
N_5(\phi_1, w_1) &= \frac{\partial \phi_1}{\partial \xi}\frac{\partial w_1}{\partial \xi}\bigg|_{\eta=0} + \left(\frac{\partial \phi_1}{\partial \xi}\right)^2, \\
N_6(w_1, \phi_1, w_2, \phi_2) &= \frac{\partial \phi_1}{\partial \xi}\frac{\partial w_2}{\partial \xi}\bigg|_{\eta=0} + \frac{\partial \phi_2}{\partial \xi}\frac{\partial w_1}{\partial \xi}\bigg|_{\eta=0} - \left(\frac{\partial \phi_1}{\partial \xi}\right)^2 \frac{\partial w_1}{\partial \eta}\bigg|_{\eta=0} + 2\frac{\partial \phi_1}{\partial \xi}\frac{\partial \phi_2}{\partial \xi}.
\end{aligned}
$$

B Solutions to the $O(\epsilon^2)$ problem

The solution to the $O(\epsilon^2)$ problem (5.11)–(5.13), subject to the conditions (5.19) and (5.20), is

$$
w_2 = \sum_{(l_1,l_2,n,m)\in S} A_{l_1}(\Re A_{l_2} + n\Im A_{l_2})e^{i(\omega_{l_1}+n\omega_{l_2})t_0}\cos(m\xi)g_{l_1,l_2,n,m}(\eta), \tag{B.1}
$$

$$
\phi_2 = \sum_{(l_1,l_2,n,m)\in S} A_{l_1}(\Re A_{l_2} + n\Im A_{l_2})e^{i(\omega_{l_1}+n\omega_{l_2})t_0}\cos(m\xi)C_{l_1,l_2,n,m}, \tag{B.2}
$$

where S is given in (5.15),

$$
\begin{aligned}
g_{l_1,l_2,n,m}(\eta) &= a_{l_1,l_2,n,m}\exp(-(1+\sqrt{1+4P_{l_1,l_2,n,m}})\frac{\eta}{2}) + \\
&\quad + \frac{1}{6}\frac{(3+i\omega_{l_1})(1+\sqrt{1+4(k_{l_1}^2+i\omega_{l_1})})(C_1)_{l_1,l_2,n,m}}{k_{l_1}^2 - m^2 - in\omega_{l_2}} \cdot \\
&\quad \cdot \exp(-(1+\sqrt{1+4(k_{l_1}^2+i\omega_{l_1})})\frac{\eta}{2}) + \\
&\quad + \frac{(1+3i)(C_2)_{l_1,l_2,n,m}}{(1-m^2) - i(\omega_{l_1}+n\omega_{l_2}-3)}\exp(-(2+i)\eta) + \\
&\quad + \frac{(C_1)_{l_1,l_2,n,m} + (C_2)_{l_1,l_2,n,m} + (C_3)_{l_1,l_2,n,m} - C_{l_1,l_2,n,m}P_{l_1,l_2,n,m}}{24P_{l_1,l_2,n,m} + (1-P_{l_1,l_2,n,m})\prod_{j=0}^{3}(n+m-j)} \cdot \\
&\quad \cdot (24 + (\eta-1)\prod_{j=0}^{3}(n+m-j))\exp(-\eta),
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
P_{l_1,l_2,n,m} &= m^2 + i(\omega_{l_1} + n\omega_{l_2}), \\
(C_1)_{l_1,l_2,n,m} &= \frac{1}{2}[(2 + l_1 l_2(2m-3))k_{l_2}^2 + in(2 - l_1 l_2)\omega_{l_2}], \\
(C_2)_{l_1,l_2,n,m} &= \frac{\sqrt{3}}{2}(l_2 - l_1)(n+m)i, \\
(C_3)_{l_1,l_2,n,m} &= \frac{1}{2}[l_1 l_2(-1)^{m/2} + \frac{1}{12}\prod_{j=0}^{3}(n+m-j)r_{l_1}],
\end{aligned}
$$

and r_i is given in (5.20). In the expression (B.3) for $g_{l_1,l_2,n,m}(\eta)$, note that $a_{l_1,l_2,n,m}$ and $C_{l_1,l_2,n,m}$ are determined uniquely by the initial conditions (5.22)–(5.23).

C Some coefficients for the $O(\epsilon^3)$ problem

Coefficients for the $O(\epsilon^3)$ problem are below for $j = 0, 1$, where $g_j(\eta) = g(\eta; i\omega_j, \nu_c)$:

$$
\begin{aligned}
\mathcal{R}_{j,1}(\eta) &= -2r_j g_j'(\eta) + i\omega_j \bar{g}_j'(\eta)(2C_{j,j,1,0} + jC_{1,1,1,2}) \\
&+ (k_j^2 - i\omega_j) g_{j,j,1,0}'(\eta) + 2(k_j^2 + i\omega_j)\Re(g_{j,j,-1,0}'(\eta)) \\
&+ k_j^2[(2C_{1,1,1,2} + 4\Re(C_{1,1,-1,2}))e^{-\eta} \\
&+ \frac{1}{2}g_1''(\eta) + \frac{1}{4}\bar{g}_1''(\eta) - \frac{3}{2}g_{1,1,1,2}'(\eta) - 3\Re(g_{1,1,-1,2}'(\eta))] \\
&+ 3ij(-\frac{1}{2}g_{1,1,1,2}'(\eta) + \Re(g_{1,1,-1,2}')(\eta)), \\
\mathcal{R}_{j,2}(\eta) &= k_{1-j}^2(C_{0,1,-1,1} + \bar{C}_{1,0,-1,1} + C_{0,1,1,1})e^{-\eta} - 2r_{1-j}g_j'(\eta) \\
&+ \left(\frac{1}{2}\right)^{1-j}\Big\{(-1)^{1-j}g_{1-j}'(\eta)[1 + i(3 - \sqrt{3})][\Re(C_{0,1,-1,1}) + (-1)^j\Im(C_{0,1,-1,1}) \\
&+ \Re(C_{1,0,-1,1}) + (-1)^{1-j}\Im(C_{1,0,-1,1})] \\
&+ \bar{g}_{1-j}'(\eta)[(-1)^{1-j} + i(\sqrt{3} + 3)]C_{0,1,1,1} \\
&+ [-1 + j + i\omega_{1-j}][\Re(g_{0,1,-1,1}'(\eta) + (-1)^j\Im(g_{0,1,-1,1}')(\eta) \\
&+ \Re(g_{1,0,-1,1}'(\eta) + (-1)^{1-j}\Im(g_{1,0,-1,1}')(\eta)] \\
&- (1 - j + i\omega_{1-j})g_{0,1,1,1}'(\eta)\Big\} + 2(k_j^2 + i\omega_j)\Re(g_{1,1,-1,0}'(\eta)), \\
\mathcal{F}_{j,1} &= \nu_c\Big[\omega_j\Big\{K''(1)|_{\nu=1/3}[\omega_j(2C_{j,j,1,0} + jC_{1,1,1,2}) - 2ir_j - \frac{i}{8}k_j^2] \\
&+ iK'''(1)|_{\nu=1/3}\omega_j^2\left(\frac{4-j}{8}\right)\Big\} - k_j^2(C_{1,1,1,2} + 2\Re(C_{1,1,-1,2}))\Big] \\
\mathcal{F}_{j,2} &= \nu_c\Big\{\left(\frac{1}{2}\right)^{1-j}\Big[(-k_{1-j}^2 + K''(1)|_{\nu=1/3}(\omega_{1-j}^2 + 3\sqrt{3}))C_{0,1,1,1} \\
&+ (-k_{1-j}^2 + K''(1)|_{\nu=1/3}(\omega_{1-j}^2 - 3\sqrt{3}))(\Re(C_{0,1,-1,1}) + (-1)^j\Im(C_{0,1,-1,1}) \\
&+ \Re(C_{1,0,-1,1}) + (-1)^{1-j}\Im(C_{1,0,-1,1}))\Big] + \frac{i}{3}\omega_j\omega_{1-j}^2K'''(1)|_{\nu=1/3} \\
&+ i\omega_j(-\frac{1}{2}k_{1-j}^2(1 + K''(1)|_{\nu=1/3}) - 2r_{1-j}K''(1)|_{\nu=1/3})\Big\} \\
\mathcal{G}_{j,1} &= k_j^2[C_{1,1,1,2}(2 + \bar{g}_1(0)) + 2\Re(C_{1,1,-1,2})(2 + g_1(0)) \\
&+ g_{1,1,1,2}(0) + 2\Re(g_{1,1,-1,2}(0)) - \frac{1}{4}(2g_1'(0) + \bar{g}_1'(0))], \\
\mathcal{G}_{j,2} &= k_j^2[(C_{0,1,-1,1} + \bar{C}_{1,0,-1,1})\left(1 + \frac{1}{2}g_1(0)\right) + C_{0,1,1,1}\left(1 + \frac{1}{2}\bar{g}_1(0)\right) \\
&+ \frac{1}{2}g_{0,1,-1,1}(0) + \frac{1}{2}\bar{g}_{1,0,-1,1}(0) + \frac{1}{2}g_{0,1,1,1}(0) - g_0'(0)],
\end{aligned}
$$

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