wrt orthogonality

from the handout on inner product and orthogonality: a set of elements $a_1, a_2, \ldots, a_n \in S$ is <u>orthogonal</u> with respect to an inner product defined on S if $\langle a_i, a_j \rangle = 0$ for $i \neq j$.

In other words, a set of elements in a vector space S is <u>orthogonal</u> with respect to an inner product defined on S if the inner product of each pair of distinct (different) elements is zero.

cos(kx), sin(kx).

The set S of functions of the form $\cos(kx)$ or $\sin(kx)$, $k \in \mathbb{Z}$, is a vector space over $\mathbb{R}$.

An inner product on the set is

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\,g(x)\,dx.$$

(standard inner product with $a_1 = -\pi$, $a_2 = \pi$).

Show the set is orthogonal with respect to the inner product: show the inner product of each pair of distinct elements is zero.

Show ① $\int_{-\pi}^{\pi} \cos kx \cos lx \, dx = 0$, $k \neq l$.

② $\int_{-\pi}^{\pi} \sin kx \sin lx \, dx = 0$, $k \neq l$.

③ $\int_{-\pi}^{\pi} \cos kx \sin lx \, dx = 0$.

① $\int_{-\pi}^{\pi} \cos kx \cdot \cos lx \, dx$

$= \int_{-\pi}^{\pi} \frac{1}{2} [\cos(kx - lx) + \cos(kx + lx)] \, dx$ trig identity

$= \frac{1}{2} \int_{-\pi}^{\pi} [\cos((k-l)x) + \cos((k+l)x)] \, dx$

$= \frac{1}{2} \left[\frac{+\sin((k-l)x)}{k-l} + \frac{\sin((k+l)x)}{k+l} \right] \Big|_{-\pi}^{\pi}$

$= \frac{1}{2} \left[\frac{1}{k-l} \sin((k-l)\pi) + \frac{1}{k+l} (\sin((k+l)\pi)) \right.$

$\left. - \left\{ \frac{1}{k-l} \sin(-(k-l)\pi) + \frac{1}{k+l} (\sin(-(k+l)\pi)) \right\} \right]$

$= 0$ ($\sin j\pi = 0$ for $j \in \mathbb{Z}$).

✓

② Can be shown the same way, using

$\sin kx \sin lx = \frac{1}{2} [\cos(kx - lx) - \cos(kx + lx)]$

③ $\displaystyle\int_{-\pi}^{\pi} \cos kx \sin lx \, dx$

$= \displaystyle\int_{-\pi}^{\pi} \tfrac{1}{2}\left[\sin(kx+lx) - \sin(kx-lx)\right]dx$ *trig identity*

$= \displaystyle\frac{1}{2}\int_{-\pi}^{\pi}\left[\sin((k+l)x) - \sin((k-l)x)\right]dx$

$= \displaystyle\frac{1}{2}\left[\frac{-\cos((k+l)x)}{k+l} + \frac{\cos((k-l)x)}{k-l}\right]\Bigg|_{-\pi}^{\pi}$ Note: assuming $k \neq l$.

$= \displaystyle\frac{1}{2}\Bigg[\frac{1}{k+l}\left(\cancel{-\cos((k+l)\pi)}\right) + \frac{1}{k-l}\left(\cancel{\cos((k-l)\pi)}\right)$

$\displaystyle + \left\{\frac{1}{k+l}\left(+\cancel{\cos((k+l)(-\pi))}\right) - \frac{1}{k-l}\cancel{\cos((k-l)(-\pi))}\right\}\Bigg]$

Use evenness of cosine:
$\cos(-y) = \cos(y)$
Distribute a $-$.

$= 0$ ✓

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(3) continued

Show $\int_{-\pi}^{\pi} \cos kx \cdot \underline{\sin kx}\, dx = 0$.

$u = \sin kx$

$du = k\,\underline{\cos kx\, dx}$

$\cos kx\, dx = \frac{1}{k}\, du$

$x = -\pi \rightarrow u = \sin k(-\pi)$

$= 0$

$x = \pi \rightarrow u = \sin k\pi$

$= 0$.

$= \frac{1}{k} \int_0^0 u\, du$

$= 0$ ✓

r. re orthogonality : e^{ikx}

The set S of functions of the form e^{ikx}, $k \in \mathbb{Z}$, is a vector space over $\mathbb{C}$. An inner product on the set is

$$\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}\,dx.$$

(standard inner product with $a_1 = 0$, $a_2 = 2\pi$).

Show each pair of distinct has an inner product of zero:

$$\int_0^{2\pi} e^{ikx}\,\overline{e^{ilx}}\,dx = 0, \quad k \neq l.$$

How do we conjugate e^{ilx}?

Let's use Euler's formula.

Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

note $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta}$

$= \cos\theta - i\sin\theta$

$= \cos(-\theta) - i\sin\theta$ because cosine is even

$= \cos(-\theta) + i\sin(-\theta)$ because sine is odd $(\sin(-y) = -\sin(y))$

$= e^{i(-\theta)}$ by Euler's formula

$= e^{-i\theta}$

$$\overline{e^{i\theta}} = e^{-i\theta}$$

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show $\int_0^{2\pi} e^{ikx} e^{-ilx}\, dx = 0$

$$\int_0^{2\pi} e^{ikx} e^{-ilx}\, dx = \int_0^{2\pi} e^{ikx-ilx}\, dx$$

$$= \int_0^{2\pi} e^{i(k-l)x}\, dx = \frac{e^{i(k-l)x}}{i(k-l)} \Big|_0^{2\pi}$$

$$= \frac{1}{i(k-l)} \left[\frac{e^{i(k-l)x}}{1} \right] \Big|_0^{2\pi}$$

$$= \frac{1}{i(k-l)} \left[e^{i(k-l)2\pi} - 1 \right]$$

$$= \frac{1}{i(k-l)} \left[\underbrace{\cos((k-l)2\pi)} + i\sin((k-l)2\pi) - 1 \right] \text{ by Euler's formula}$$

$$= \frac{1}{i(k-l)} \left[1 + 0 - 1 \right] = 0 \checkmark$$

(1) We have shown that the set of fncs

$\sin kx, \cos kx$ is orthogonal ($k \in \mathbb{Z}$) $\longleftarrow$ "harmonics"

with respect to the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$$

and the set of functions

e^{ikx}, $k \in \mathbb{Z}$ is orthogonal $\longleftarrow$ "harmonics"

with respect to the inner product

$$\langle f, g \rangle = \int_{0}^{2\pi} f(x) \overline{g(x)}\, dx.$$

(2) These sets are also <u>complete</u>: no function can be orthogonal to all harmonics.

(3) They are convenient. Other sets are orthogonal & complete, but they are more complicated to work with.