CN33: Laplace's Eq. on a disk: example

ex Solve the BVP

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0, \quad \begin{array}{c}-\pi\le\theta\le\pi\\ 0\le r\le 1\end{array}$$

$$u(1,\theta)=\theta$$

The general solution to the PDE is

$$u(r,\theta)=\underbrace{a_0}+\underbrace{a_1}r\cos\theta+b_1 r\sin\theta+\underbrace{a_2}r^2\cos 2\theta+b_2 r^2\sin 2\theta+\cdots$$

Impose the BC:

$$u(1,\theta)=a_0+a_1\cos\theta+b_1\sin\theta+a_2\cos 2\theta+b_2\sin 2\theta+\cdots$$
$$=\theta$$

We previously expanded $f(x)=x$, $-\pi\le x\le\pi$ into a F.S. using

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}x\,dx=0$$

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos kx\,dx=0, \quad k=1,2,\ldots$$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin kx\,dx=\frac{-2}{k}(-1)^k, \quad k=1,2,\ldots$$

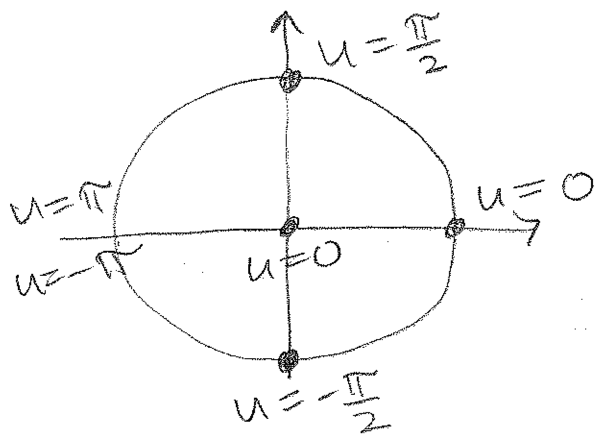
Got $x=2\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\cdots\right)$.

$a_0 = 0$, $a_1 = 0$, $a_2 = 0$, ...

$b_k = \frac{-2}{k}(-1)^k$: $b_1 = 2$, $b_2 = -1$, $b_3 = \frac{2}{3}$, ...

The solution to the BVP is

$$u(r,\theta) = 2r\sin\theta - r^2\sin 2\theta + \frac{2}{3}r^3\sin 3\theta - \cdots$$



On the circle u has a sudden jump at $\theta = \pi$.

Inside the circle that jump is gone.

Laplace's eq has smooth solutions.