N28 We will see that we can expand $f(x)$ into a sum of constant multiples of harmonic fncs (a linear combination of harmonic functions). | Fourier series

We want to consider $f(x)$ a very general function defined on $0 \le x < 2\pi$. We'll write the Fourier series (F.S.)

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \cdots + c_{-2} e^{-i2x} + c_{-1} e^{-ix} + c_0 + c_1 e^{ix} + c_2 e^{2ix} + \cdots$$

An expansion into $a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$ is equivalent.

One can also consider a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} c(k) e^{ikx}\, dk \,, \qquad -\infty < x < \infty .$$

Here, note $k \in \mathbb{R}$.

A discrete Fourier series is defined at only n x-values, and it only needs n frequencies

$$f(x) = c_0 + c_1 e^{ix} + c_2 e^{i2x} + \cdots + c_n e^{i(n-1)x} ,$$

the eq. holds at n equally spaced points $x = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots$ below 2π.

Fourier coefficients

Suppose $f(x)$ is a func. with period 2π. Its graph from 2π to 4π is a repitition of its graph from 0 to 2π. The function repeats like that along the whole real line:

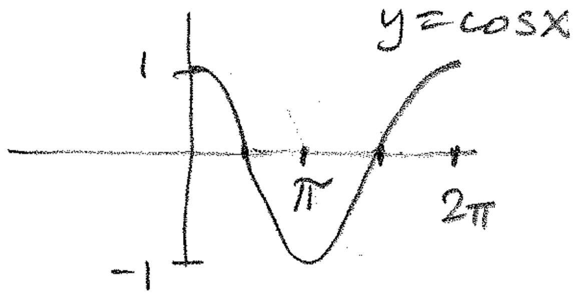
$$f(x+2\pi) = f(x) \text{ for every } x.$$

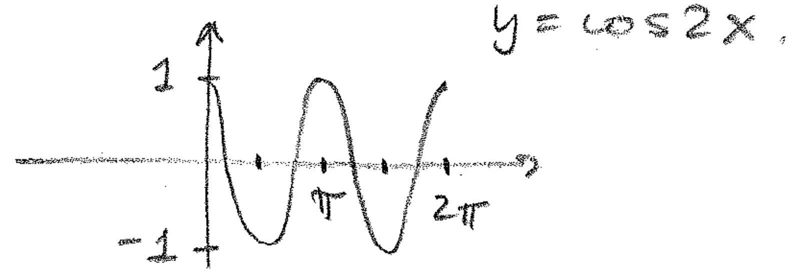
$\cos x, \sin x$ have period 2π.

a constant func has period 2π.

$\cos 2x, \sin 2x$ oscillate twice as fast as $\cos x, \sin x$.

They have period 2π and also period π.





The idea of a Fourier Series (F.S.) is to expand every 2π-periodic func $f(x)$ into a linear combination of harmonics:

$$f(x) = a_0 + a_1\cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

all the terms on the right are 2π-periodic. The inner product formula could have involved an integral over any interval of length 2π. We used $-\pi$ to π.

Find a_k, $k = 1, 2, 3, \ldots$ & b_k, $k = 1, 2, 3, \ldots$

Mult. the F.S. by $\cos kx$ and integrate from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \int_{-\pi}^{\pi} (a_0 + a_1\cos x + b_1 \sin x + \cdots + a_k \cos kx + b_k \sin kx + \cdots)\cos kx\,dx$$

$$= \cancelto{0}{\int_{-\pi}^{\pi} a_0 \cos kx\,dx} + \cancelto{0}{\int_{-\pi}^{\pi} a_1 \cos x \cos kx\,dx}$$

$$+ \cancelto{0}{\int_{-\pi}^{\pi} b_1 \sin x \cos kx\,dx} + \cdots$$

$$+ \int_{-\pi}^{\pi} a_k \cos kx \cdot \cos kx\,dx +$$

$$+ \cancelto{0}{\int_{-\pi}^{\pi} b_k \sin kx \cdot \cos kx\,dx} + \cdots$$

Most of the integrals are 0 because of orthogonality of harmonics.

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \int_{-\pi}^{\pi} a_k \cos^2 kx\,dx$$

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = a_k \int_{-\pi}^{\pi} \frac{1+\cos 2kx}{2}\,dx \quad \text{(trig identity)}$$

$$= \frac{1}{2} a_k \left[x + \frac{\sin 2kx}{2k} \right]\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2} a_k \left[\pi + 0 - (-\pi + 0) \right]$$

$$= \frac{1}{2} a_k (\pi + \pi)$$

$$= \frac{1}{\cancel{2}} a_k \cdot \cancel{2}\pi$$

$$= a_k \pi$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos kx\,dx \quad (k \neq 0)$$

Similar argument gives

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin kx\,dx, \quad k = 1, 2, 3, \ldots$$

Find a_0 in $f(x) = a_0 + a_1 \cos x + b_1 \sin x + \dots + a_k \cos kx + b_k \sin kx + \dots$

mult the F.S. by 1 and integrate from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} (a_0 + a_1 \cos x + b_1 \sin x + \dots + a_k \cos kx + b_k \sin kx)\,dx$$

$$= \int_{-\pi}^{\pi} a_0\,dx + \underbrace{\cancel{\int_{-\pi}^{\pi} a_1 \cos x\,dx}}_{0} + \underbrace{\cancel{\int_{-\pi}^{\pi} b_1 \sin x\,dx}}_{0} + \dots + \underbrace{\cancel{\int_{-\pi}^{\pi} a_k \cos kx\,dx}}_{0} + \underbrace{\cancel{\int_{-\pi}^{\pi} b_k \sin kx\,dx}}_{0} + \dots$$

most of the integrals are 0 because of orthogonality of $\cos 0x$ with the other harmonics.

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx$$

$$= a_0 \int_{-\pi}^{\pi} dx$$

$$= a_0 x \Big|_{-\pi}^{\pi} = a_0(\pi - (-\pi)) = a_0 \cdot 2\pi$$

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0 \longrightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad \longleftarrow \text{avg value of } f(x).$$

The F.S. of a 2π-periodic fnc is

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

with
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \quad , k = 1, 2, 3, \ldots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx \quad , k = 1, 2, 3, \ldots$$

Each term in the series is a projection of f onto one of the harmonics.

Equivalently $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$

with $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx$.