specific parameters nonhomogeneous DE

N8: Ex2. $-cu'' + qu = f_0$
$u(0)=0, \; u(1)=0$ } BVP.

solution $u(x) = ?$

$-cu''+qu = 0$ was a <u>homogeneous eq</u>:
all terms depend on u.
RHS is 0.

$-cu''+qu = f_0$, $f_0 \neq 0$ is a <u>nonhomogeneous eq</u>
f_0 is a <u>nonhomogeneous term</u>.

The solution to the nonhomogeneous ODE has the form $u(x) = u_h(x) + u_p(x)$, where

$u_h(x)$ is the "homogeneous solution":
satisfies $-cu''+qu=0$.
We found $u_h(x) = d_1 e^{\sqrt{q/c}\,x} + d_2 e^{-\sqrt{\frac{q}{c}}\,x}$.

$u_p(x)$ is a <u>particular</u> solution that
satisfies $-cu''+qu = f_0$.

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Because the nonhomogeneous term f_0 is a constant, we will try to find a constant particular solution of the form $u_p(x) = P$: substitute $u(x) = u_p(x) = P$ into the nonhomog. DE $-cu'' + qu = f_0$ to determine P:

$$u_p = P \rightarrow u_p' = 0 \rightarrow u_p'' = 0$$

$$-cu_p'' + qu_p = f_0 \rightarrow 0 + qP = f_0 \rightarrow P = \frac{f_0}{q}.$$

so $u(x) = d_1 e^{\sqrt{\frac{q}{c}}x} + d_2 e^{-\sqrt{\frac{q}{c}}x} + \frac{f_0}{q}$.

now apply BCs: (1) $u(0) = 0$, (2) $u(1) = 0$.

$$\left.\begin{aligned} (1)&: u(0) = d_1 + d_2 + \frac{f_0}{q} = 0 \\ (2)&: u(1) = d_1 e^{\sqrt{q/c}} + d_2 e^{-\sqrt{q/c}} + \frac{f_0}{q} = 0 \end{aligned}\right\}$$

2 eqns for 2 unknowns d_1, d_2.

(1) $\rightarrow$ $d_2 = -\frac{f_0}{q} - d_1$ sub. into (2):

(3) $d_1 e^{\sqrt{q/c}} + \left(-\frac{f_0}{q} - d_1\right) e^{-\sqrt{q/c}} + \frac{f_0}{q} = 0$

collect d_1 terms, and solve for d_1.

$$\boxed{d_1 \left(\cancel{\frac{e^{\sqrt{q/c}} - e^{-\sqrt{q/c}}}{e^{\sqrt{q/c}} - e^{-\sqrt{q/c}}}}\right) = \frac{-\frac{f_0}{q} + \frac{f_0}{q} e^{-\sqrt{q/c}}}{e^{\sqrt{q/c}} - e^{-\sqrt{q/c}}}} = -\frac{f_0}{q}\left[\frac{1 - e^{-\sqrt{q/c}}}{e^{\sqrt{q/c}} - e^{-\sqrt{q/c}}}\right]$$

$$d_2 = -\frac{f_0}{q} - d_1 \rightarrow \boxed{d_2 = -\frac{f_0}{q}\left[1 + \frac{-1 + e^{-\sqrt{q/c}}}{e^{\sqrt{q/c}} - e^{-\sqrt{q/c}}}\right]}$$

We've solved $-cu'' + qu = f_0$, $u(0) = 0$, $u(1) = 0$:

$$\boxed{u(x) = d_1 e^{\sqrt{q/c}\,x} + d_2 e^{-\sqrt{q/c}\,x} + \frac{f_0}{q}}$$

where d_1 & d_2 are above.

Ex 3 $-cu'' + qu = f_0 e^{bx}$ ← specified nonhomog.

Proceed as before, except seek particular solution $u_p(x)$ of form $u_p(x) = P e^{bx}$. (Find P.)

Ex 4 $-cu'' + qu = f_0 \sin bx$.

Seek $u_p(x) = P_1 \sin bx + P_2 \cos bx$.

Find P_1 & P_2.

Ex 5 $-cu'' + qu = f_0 \cos bx$

Seek $u_p(x)$ as in Ex 4.

Extensions:

The solution to our ODE minimizes a potential-energy integral. One studies such problems in <u>calculus of variations</u>.

If a load $f(x)$ is applied to a beam with bending stiffness $c(x)$, the deflection $u(x)$ satisfies

$$\frac{d^2}{dx^2}\left(c \frac{d^2 u}{dx^2} \right) = f(x).$$

This ODE is <u>fourth</u> -order.

We impose 2 BCs at each end.