An elastic bar:

an introductory differential eq. (ODE)

$$-\frac{d}{dx}\left(c\,\frac{du}{dx}\right) = f \qquad (p156)$$

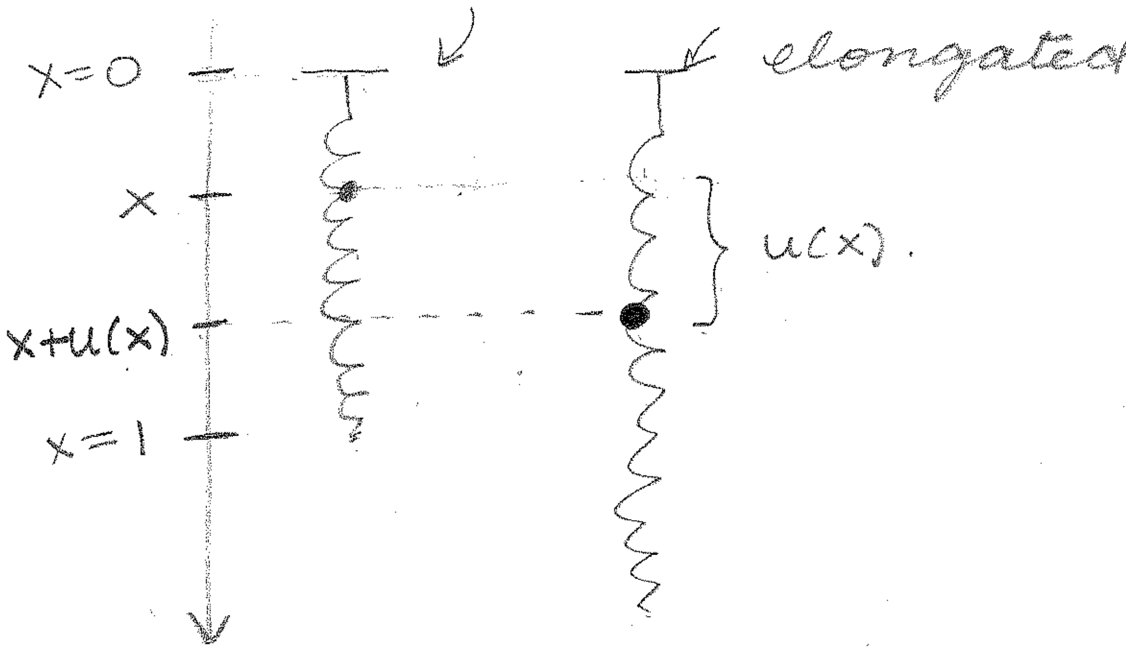
c = given parameter (constant)

$F(x)$ = given function

$u(x)$ = ?

This is an ODE.
Let's derive how it models a 1-D elastic bar in equilibrium.

Elastic bar at rest p 155



Displacement is $u(x)$.

The elongation in the bar is

$$e = \frac{du}{dx}$$

Recall $\frac{dy}{dx}$ in calculus gives the rate of change of y with respect to x.

units on $\frac{dy}{dx} = \left[\frac{dy}{dx}\right] = \frac{\text{units of } y}{\text{units of } x}$

$$= \frac{[y]}{[x]}$$

Standard basic units are units of

length l.

time t.

mass m.

Temperature T.

We can combine these to get the units of other quantities

area l^2

volume l^3

Velocity $\frac{l}{t}$

density $\frac{m}{vol} = \frac{m}{l^3}$

acceleration $\frac{l}{t^2}$

force. $\frac{ml}{t^2}$

rate $\frac{1}{t}$ "per time"

note for elongation (strain)

$$[e] = \left[\frac{du}{dx}\right] = \frac{[u]}{[x]} = \frac{l}{l} = 1$$

elongation is a dimensionless variable.

Stretching produces an internal force w proportional to the strain $\frac{du}{dx}$: $w = c\frac{du}{dx}$

In symbols, $w \propto \frac{du}{dx}$

*$u=0$ at fixed end : $u(0) = 0$

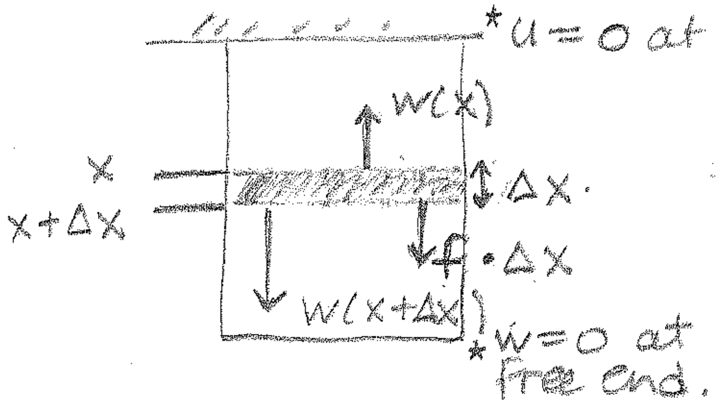


Fig 3.1 p156

equilibrium of an elastic bar

f is the external force per unit length

In equilibrium all the forces balance.

$$w(x+\Delta x) - w(x) + f \cdot \Delta x = 0$$

$$\frac{\left(c\frac{du}{dx}\right)(x+\Delta x) - \left(c\frac{du}{dx}\right)(x) + f \cdot \Delta x = 0}{\Delta x}$$

$$\frac{d}{dx}\left(c\frac{du}{dx}\right) = -f, \qquad \text{as } \Delta x \to 0.$$

$$0 \le x \le 1.$$

c, f are specified

$u(x) = ?$

Can we solve $\boxed{-\frac{d}{dx}\left(c\frac{du}{dx}\right) = f}$ DE

for $u(x)$, subject to two <u>boundary conditions</u> (BCs):

top end fixed $\rightarrow$ $u(0) = 0$

$w(1) = 0$: $\rightarrow$ $\left(c\frac{du}{dx}\right)(1) = 0$ BCs.

(no strain at free end)

How can we solve this <u>boundary-value problem</u> (BVP)

(= DE + BCs) ?

Elastic bar, 6.

$$-\frac{d}{dx}\left(c\,\frac{du}{dx}\right) = {}^{+}f \quad , \; c, f \text{ constant}$$

Integrate: $-c\,\dfrac{du}{dx} = fx + c_1$ (**)

Integrate: $-cu = f\dfrac{x^2}{2} + \boxed{c_1}x + c_2$ (*)

Impose the boundary conditions,

subst $x=0$ into (*):

① $u(0)=0$: $-c\,u(0) = f\cdot\dfrac{0^2}{2} + c_1\cdot 0 + c_2$

$$0 = c_2$$

Now I know $-cu = f\dfrac{x^2}{2} + c_1 x$

substitute $x=1$ into (**)

② $\left(c\,\dfrac{du}{dx}\right)(1) = 0$: $-\left(c\,\dfrac{du}{dx}\right)(1) = f\cdot 1 + c_1$

$0 = f + c_1 \rightarrow c_1 = \boxed{-f}$ Now I know:

$$-cu = \frac{fx^2}{2} - fx \rightarrow u(x) = \frac{fx\left(\frac{x}{2}-1\right)}{-c} \Rightarrow$$

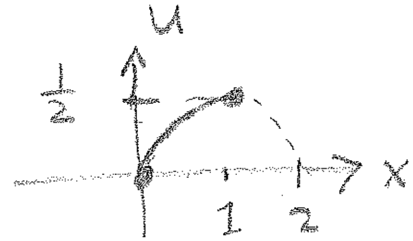
$$\boxed{u(x) = \frac{fx\left(1-\frac{x}{2}\right)}{c}}, \; 0 \le x \le 1.$$

Verify the solution to the boundary-value problem (BVP) satisfies the ODE and BCs (by substitution).

Think about

Ex 1: $c = 1$, $f = 1$.

$u(x) = x\left(1 - \frac{x}{2}\right)$.



Ex 2: $c = 1$, $f = 1$.

$u(0) = 0$, $\underline{u(1) = b}$

Solve!